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Analysis of Quintic Equations.

BY EMORY MCCLINTOCK, LL.D., F.I.A.

1. The solution of equations of degrees below the fifth depends in each case on that of an auxiliary equation, or resolvent, one degree lower than the given equation. On applying the same principles towards the solution of the general quintic, its resolvent is found to be of the sixth degree instead of the fourth, for reasons which it is not necessary to recite. Abandoning therefore the idea of a solution, mathematicians have nevertheless considered it an interesting problem to accomplish in the simplest manner what is known as the resolution of the quintic. This has been defined by Sir James Cockle* as "the expression of its roots in terms of those of its resolvent sextic." Assuming that a sextic is discovered, appropriate for the purpose, the object to be attained is the expression of the roots of the quintic as a function of those of the sextic. Methods of resolution may be divided into two classes: those in which the usual theory of equations is illustrated by the intervenience of four auxiliary quantities which I shall herein call the elements, and those in which it is not. In methods of the first class each root is regarded as a sum of elements affected by powers of roots of unity as coefficients, and the problem consists in finding the simplest expressions for the elements as functions of the roots of the resolvent. In methods of the second class the intervening elements are omitted. If the chief object of effecting the resolution be considered, as I think it must, to be the illustration of the general theory of equations in the case of the lowest degree for which algebraic solution is impossible, the methods of the first class will be preferred. Before closing, I shall present one or two methods of the second class; but my leading purpose, apart from the indication of suitable resolvents, is to exhibit the elements most directly as functions of the roots of resolvents, by what I shall for brevity call element-formulæ.

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* "On the Resolution of Quintics," Quarterly Journal, June, 1860.

2. The only resolvent hitherto* known is that of Malfatti. It has been independently rediscovered by several analysts, and has been shown in various different forms. The simplest rational function of the roots of the quintic which can be determined by Malfatti's resolvent is a quantity which I denote by v. The importance of this quantity has, I think very singularly, been wholly overlooked. I find that, analogous to Malfatti's resolvent, which I call the dexter resolvent, there is another which I call the sinister resolvent, and a third, so to speak between the two, and of greater importance as well as greater analytic symmetry than either, which I call the central resolvent. The simplest rational functions of the roots determined by the sinister and central resolvents, respectively, I denote by s and t. Between the three auxiliary quantities just mentioned exists the relation $s = t^2v$. I shall state what seem to be the most direct formulæ for expressing t and v as rational functions of each other, and indicate means for similarly expressing s and v. Whenever, therefore, one of the three auxiliary quantities is found by means of one of the three resolvents, the others become known. The element-formulæ to be shown in the next paragraph contain only the auxiliary quantities and the coefficients of the quintic, and are thus available whichever of the three resolvents is employed. Beginning with a summary statement of what appears the best attainable method of resolution, with illustrations, I shall proceed to give a sketch of the history of the subject, in which I shall endeavor to give due credit to each discoverer. Entering then upon a more general discussion, I shall develop the theory of the three resolvents, followed by that of the element-formulæ, and of the expression as functions of each other of t and v. After treating of various subsidiary matters, I shall conclude with a number of suggestions relating to methods of resolution of the second class, and to solutions of the quintic by means of tabulated functions.

3. If the general equation of the fifth degree be

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \tag{1}$$

let

$$\left.\begin{aligned} \gamma &= a^{-2}(ac - b^2), \\ \delta &= a^{-3}(a^2d - 3abc + 2b^3), \\ \varepsilon &= a^{-4}(a^3e - 4a^2bd + 6ab^2c - 3b^4), \\ \zeta &= a^{-5}(a^4f - 5a^3be + 10a^2b^2d - 10ab^3c + 4b^5). \end{aligned}\right\} \tag{2}$$

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* Except as given in a recent paper by the present writer, "On the Resolution of Equations of the Fifth Degree" (*American Journal of Mathematics*, 1884, Vol. VI, pp. 301–315). Such parts of that paper as appear to possess permanent value will be embodied herein as original, with footnote references to "Vol. VI." I take this opportunity to point out certain *errata*. Page 302, line 2, for p^5 read p_5. Page 310, line 12, for "and ϕ" read "when ϕ is, and sometimes when ϕ". Page 311, last line, for $-2\delta\varepsilon\zeta$ read $+2\delta\varepsilon\zeta$.

Let there be an auxiliary quintic, $\mathrm{A}_y = t^5 + 10\gamma t^3 + 10\delta t^2 + 5\varepsilon t + \zeta$, with its canonizant, $\mathrm{C}_y = c_0 t^3 + c_1 t^2 + c_2 t + c_3$, and its simplest linear covariant, $\mathrm{L}_y = l_0 t + l_1$, where

$$\begin{aligned}
c_0 &= -\gamma^3 + \gamma\varepsilon - \delta^2,\\
c_1 &= -\gamma^2\delta + \gamma\zeta - \delta\varepsilon,\\
c_2 &= -\gamma\delta^2 + \gamma^2\varepsilon + \delta\zeta - \varepsilon^2,\\
c_3 &= \quad 2\gamma\delta\varepsilon - \gamma^2\zeta - \delta^3,\\
l_0 &= -15\gamma^4\varepsilon + 10\gamma^3\delta^2 - 2\gamma^2\delta\zeta + 14\gamma^2\varepsilon^2 - 22\gamma\delta^2\varepsilon + \gamma\zeta^2 + 9\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3,\\
l_1 &= 9\gamma^4\zeta - 20\gamma^3\delta\varepsilon + 10\gamma^2\delta^3 + 8\gamma^2\varepsilon\zeta - 12\gamma\delta\varepsilon^2 - 2\gamma\delta^2\zeta + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta.
\end{aligned}$$

Let a value of t be found by the numerical solution of the central resolvent*

$$\mathrm{A}_y \mathrm{L}_y - 25\mathrm{C}_y^2 = 0, \tag{3}$$

which may be written thus:

$$\left.\begin{aligned}
&(l_0 - 25c_0^2)\,t^6 + (l_1 - 50c_0c_1)\,t^5 + 5\,(2\gamma l_0 - 5c_1^2 - 10c_0c_2)\,t^4\\
&+ 10\,(\gamma l_1 + \delta l_0 - 5c_0c_3 - 5c_1c_2)\,t^3 + 5\,(2\delta l_1 + \varepsilon l_0 - 5c_2^2 - 10c_1c_3)\,t^2\\
&+ (5\varepsilon l_1 + \zeta l_0 - 50c_2c_3)\,t + \zeta l_1 - 25c_3^2 = 0.
\end{aligned}\right\} \tag{4}$$

Also, after t is known, let v be found by using either of these expressions:

$$v = -\mathrm{C}_y\mathrm{A}_y^{-1} = -\tfrac{1}{25}\,\mathrm{L}_y\mathrm{C}_y^{-1}. \tag{5}$$

Then the five roots of (1), say x_1, x_2, x_3, x_4, x_5, are severally determined by the equation†

$$x_{r+1} = \omega^{4r}u_1 + \omega^{3r}u_2 + \omega^{2r}u_3 + \omega^r u_4 - ba^{-1}, \tag{6}$$

where ω is any unreal fifth root of unity, and r is 0, 1, 2, 3, or 4; and where u_1, u_2, u_3, u_4, are determined by the element-formulæ

$$\left.\begin{aligned}
u_1^5 &= \tfrac14 r_1 + \tfrac14 r_2 + \surd(s_1 + s_2),\\
u_2^5 &= \tfrac14 r_1 - \tfrac14 r_2 + \surd(s_1 - s_2),\\
u_3^5 &= \tfrac14 r_1 - \tfrac14 r_2 - \surd(s_1 - s_2),\\
u_4^5 &= \tfrac14 r_1 + \tfrac14 r_2 - \surd(s_1 + s_2),
\end{aligned}\right\} \tag{7}$$

where

$$r_1 = -\zeta - 20tv, \tag{8}$$

$$\left.\begin{aligned}
r_2 = (\gamma^2 - v)^{-1}\,&(12\gamma tv^2 - \delta v^2 - t^3v^2 + 4\gamma^3tv + 2\gamma^2\delta v\\
&+ \gamma\delta t^2v + \delta^2tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac12},
\end{aligned}\right\} \tag{9}$$

$$s_1 = \tfrac{1}{16}r_1^2 + \tfrac{1}{16}r_2^2 + \gamma^5 + 10\gamma^3v + 5\gamma v^2, \tag{10}$$

$$s_2 = \tfrac18 r_1r_2 - (5\gamma^4 + 10\gamma^2v + v^2)\,v^{\frac12}. \tag{11}$$

The result is unaffected by any change in the signs given to the square-roots.

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*Vol. VI, p. 315. The quantity t was there taken with the contrary sign. Its existence and properties appear to have remained unnoticed by previous writers.

† According to Euler's well-known theory; the quantities u_1, u_2, u_3, u_4, remaining to be ascertained.

4. As an illustration let us take the equation

$$x^5 - 5x^4 - 25x^3 + 125x^2 - 36x - 180 = 0. \tag{12}$$

Here $a = 1$, $b = -1$, $c = -\frac{5}{2}$, $d = \frac{25}{2}$, $e = -\frac{36}{5}$, $f = -180$. Hence, by (2), $\gamma = -\frac{7}{2}$, $\delta = 3$, $\varepsilon = \frac{124}{5}$, $\zeta = -120$. We have therefore $c_0 = -\frac{2117}{40}$, $c_1 = \frac{6177}{20}$, $c_2 = -\frac{31987}{50}$, $c_3 = \frac{4611}{5}$, $l_0 = \frac{6905219}{125}$, $l_1 = -\frac{4891314}{25}$, and when these values are substituted in (4), and the result cleared of fractions, the resolvent is

$$\left.\begin{aligned} &-118277109t^6 + 4973134020t^5 - 48088650860t^4 + 166597190880t^3 \\ &-187938837936t^2 - 111321302240t + 17735889600 = 0. \end{aligned}\right\} \tag{13}$$

One root of this is $t = 30$. Using this value in (5), we find that $v = \frac{1}{20}$. The other values of t, with the corresponding values of v, are

t	6	$\frac{130}{51}$	$\frac{66}{19}$	$\frac{22}{69}$	$-\frac{18}{61}$
v	$\frac{841}{500}$	$\frac{2601}{500}$	$\frac{361}{500}$	$\frac{4761}{500}$	$\frac{3721}{500}$

Taking $t = 30$, $v = \frac{1}{20}$, we have $r_1 = 90$, $r_2 = 6\sqrt{5}$, $s_1 = -\frac{146}{5}$, $s_2 = -\frac{1018}{125}\sqrt{5}$, so that by (7)

$$\left.\begin{aligned} u_1^5 &= \tfrac{45}{2} + \tfrac{3}{2}\sqrt{5} + \sqrt{(-\tfrac{146}{5} - \tfrac{1018}{125}\sqrt{5})}, \\ u_2^5 &= \tfrac{45}{2} - \tfrac{3}{2}\sqrt{5} + \sqrt{(-\tfrac{146}{5} + \tfrac{1018}{125}\sqrt{5})}, \\ u_3^5 &= \tfrac{45}{2} - \tfrac{3}{2}\sqrt{5} - \sqrt{(-\tfrac{146}{5} + \tfrac{1018}{125}\sqrt{5})}, \\ u_4^5 &= \tfrac{45}{2} + \tfrac{3}{2}\sqrt{5} - \sqrt{(-\tfrac{146}{5} - \tfrac{1018}{125}\sqrt{5})}. \end{aligned}\right\} \tag{14}$$

The arithmetical evaluation of these quantities is not possible, since they are imaginary. There are five real roots in the example selected, and the same difficulty arises as in the corresponding ("irreducible") case in cubic equations, where there are three real roots. Yet we have here the correct analytic expression for the elements of the roots.

5. As another example—this time with but one real root—let

$$x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5 = 0. \tag{15}$$

Here $a = 1$, $b = 1$, $c = \frac{9}{10}$, $d = \frac{1}{2}$, $e = -\frac{4}{5}$, $f = -5$, so that, by (2), $\gamma = -\frac{1}{10}$, $\delta = -\frac{1}{5}$, $\varepsilon = -\frac{2}{5}$, $\zeta = -1$; whence $c_0 = .001$, $c_1 = .022$, $c_2 = .04$, $c_3 = .002$, $l_0 = -.0062$, $l_1 = -.0353$, and by (4), after multiplying by 2500,

$$-15\tfrac{9}{16}t^6 - 91t^5 - 19\tfrac{3}{4}t^4 + 9t^3 + 102t^2 + 182t + 88 = 0. \tag{16}$$

One root is $t = \frac{4}{3}$, and this substituted in (5) gives $v = .018$. Hence $r_1 = .52$, $r_2 = .104\sqrt{5}$, $s_1 = .019928$, $s_2 = .00660256\sqrt{5}$, and by (7)

$$\left.\begin{aligned} u_1^5 &= .13 + .026\sqrt{5} + \sqrt{(.019928 + .00660256\sqrt{5})}, \\ u_2^5 &= .13 - .026\sqrt{5} + \sqrt{(.019928 - .00660256\sqrt{5})}, \\ u_3^5 &= .13 - .026\sqrt{5} - \sqrt{(.019928 - .00660256\sqrt{5})}, \\ u_4^5 &= .13 + .026\sqrt{5} - \sqrt{(.019928 + .00660256\sqrt{5})}, \end{aligned}\right\} \tag{17}$$

and from (6), putting $r = 0$, we have as the real root of the quintic,

$$\left.\begin{aligned} x_1 &= u_1 + u_2 + u_3 + u_4 - 1 \\ &= 0.82161 + 0.67843 - 0.05036 + 0.28501 - 1 = 0.73469. \end{aligned}\right\} \qquad (18)$$

6. It was proved by Lagrange that in determining the elements the extraction of more than one fifth-root is unnecessary. We shall see later that $u_1u_4 = -\gamma + v^{\frac{1}{2}}$, and $u_2u_3 = -\gamma - v^{\frac{1}{2}}$, so that u_4 and u_3 are readily found when u_1 and u_2 are known. For finding u_2 we may note that $u_2 = (u_2u_3)\, u_1^2 (u_1^2 u_3)^{-1}$, and we shall see that $u_1^2 u_3 = -\frac{1}{2}\delta - \frac{1}{2} t v^{\frac{1}{2}} + \surd(s_3 + s_4)$, where $s_3 = \gamma^3 - \gamma v + \frac{1}{4}\delta^2 + \frac{1}{4} t^2 v$, and $s_4 = (v - \gamma^2 + \frac{1}{2}\delta t)\, v^{\frac{1}{2}}$. No perceptible benefit is derived, however, from the latter variation. To illustrate the determination of u_3 and u_4 when u_2 and u_1 are known, let us take the example in the preceding paragraph. Here $\gamma = -.1$, $v^{\frac{1}{2}} = .13416$, $u_1u_4 = .23416$, $u_2u_3 = -.03416$, $u_1 = .82161$, $u_2 = .67843$; hence $u_4 = .28501$, $u_3 = -.05036$, as before.

7. The foregoing method of resolution is available for all equations of the fifth degree wherein (5) is not indeterminate, except those in which $v = \gamma^2$ and those in which $v = 0$. In the former case, $u_1u_2u_3u_4 = 0$, or say $u_4 = 0$, since the final result is the same when one element vanishes, whichever be chosen. In the latter case, $u_1u_4 = u_2u_3$. The quantity $u_1u_2u_3u_4$ was called "the resolvent product" by Sir James Cockle, who pointed out* that all quintics are solvable in which the resolvent product vanishes, and furnished a criterion by which to recognize a quintic of this class when met with. The following is a modification† of the criterion in question: $v = \gamma^2$ when

$$\left.\begin{aligned} &[1600\gamma^6 - 640\gamma^4\varepsilon + 160\gamma^3\delta^2 - 52\gamma^2\delta\zeta + 64\gamma^2\varepsilon^2 + \gamma\zeta^2 \\ &\qquad + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 = \gamma^2\mathrm{D}, \end{aligned}\right\} \qquad (19)$$

where D = discriminant × $a^{-8} = 8\,[432\gamma^5\zeta^2 - 1440\gamma^4\delta\varepsilon\zeta + 800\gamma^4\varepsilon^3 + 640\gamma^3\delta^3\zeta - 400\gamma^3\delta^2\varepsilon^2 - 180\gamma^3\varepsilon\zeta^2 + 330\gamma^2\delta^2\zeta^2 + 560\gamma^2\delta\varepsilon^2\zeta - 320\gamma^2\varepsilon^4 - 1260\gamma\delta^3\varepsilon\zeta + 720\gamma\delta^2\varepsilon^3 - 15\gamma\delta\zeta^3 + 20\gamma\varepsilon^2\zeta^2 + 432\delta^5\zeta - 270\delta^4\varepsilon^2 + 45\delta^2\varepsilon\zeta^2 - 80\delta\varepsilon^3\zeta + 32\varepsilon^5 + \frac{1}{8}\zeta^4]$. In any such case, one or two critical cases excepted, the process of solution about to be described will no doubt be found satisfactory. When $v = \gamma^2$,

$$t = \frac{800\gamma^5\delta + 16\gamma^4\zeta + 920\gamma^3\delta\varepsilon - 560\gamma^2\delta^3 - 8\gamma^2\varepsilon\zeta - 38\gamma\delta\varepsilon^2 - 48\gamma\delta^2\zeta + 44\delta^3\varepsilon - \delta\zeta^2 + \varepsilon^2\zeta}{-10400\gamma^6 + 1960\gamma^4\varepsilon - 240\gamma^3\delta^2 + 48\gamma^2\delta\zeta - 86\gamma^2\varepsilon^2 + 28\gamma\delta^2\varepsilon + \gamma\zeta^2 - 16\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3}. \qquad (20)$$

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* Appendix to Lady's and Gentleman's Diary, 1858, p. 82, as quoted by Harley, Manchester Memoirs, Vol. XV. Mr. Harley gave a convenient process of solution, but different from that now presented, and more complex.

† Vol. VI, p. 312.

Supposing $u_4 = 0$, we have $v^{\frac{1}{5}} = \gamma$, and $u_2 u_3 = -2\gamma$, giving u_2 when u_3 is known. Then

$$u_1^5 = \tfrac{1}{4}\gamma^{-2}(\gamma t + \delta)^2(\gamma t - \delta) = \tfrac{1}{4}(\gamma t^3 + \delta t^2 - \gamma^{-1}\delta^2 t - \gamma^{-2}\delta^3), \tag{21}$$

$$u_3 = -u_1^{-2}(\gamma t + \delta). \tag{22}$$

Take for example this equation,

$$x^5 + 5x^3 + 5x^2 + 5x - \tfrac{35}{2} = 0. \tag{23}$$

Here $\gamma = \frac{1}{2}$, $\delta = \frac{1}{2}$, $\varepsilon = 1$, $\zeta = -\frac{35}{2}$, and by (20) $t = 3$, so that $u_1 = 4^{\frac{1}{5}}$, $u_3 = -2^{\frac{1}{5}}$, $u_2 = 2^{-\frac{1}{5}}$, $u_4 = 0$, and hence a root of (23), the only real root, is $1.31951 + 0.87055 - 1.14870 = 1.04136$. Among the exceptional or critical cases are that of De Moivre, wherein $\delta = 0$ and $\varepsilon = 4\gamma^2$, and that of Euler, wherein $\gamma = 0$ and $16\delta^4 + 2\delta\varepsilon\zeta - \varepsilon^3 = 0$, of which solutions are well known.

8. The second class of quintics spoken of in the last paragraph as requiring special treatment is that wherein $u_1 u_4 = u_2 u_3$, a class first separately discussed, so far as my knowledge goes, by Mr. G. P. Young,[*] who noted that within it are included, with others, such quintics as might be employed, under Gauss's theory, in the determination of roots of unity. I gave somewhat later[†] a method of solution for this class of cases, less circuitous than that indicated by Mr. Young, but much more so than that which will now be presented. I also supplied a criterion[‡] by which to recognize such cases when they occur, as follows: $u_1 u_4 = u_2 u_3$ when $v = 0$, and $v = 0$ when

$$25\gamma^6 - 35\gamma^4\varepsilon + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon + 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3 = 0. \tag{24}$$

In any such case, not critical, the quintic may be solved instantly by employing the normal formulæ (7) for the elements, but putting $r_1 = -\zeta$, $r_2 = (3\gamma^4 - 4\gamma^2\varepsilon + 4\gamma\delta^2 - \delta\zeta + \varepsilon^2)(\gamma^3 - \gamma\varepsilon + \delta^2)^{-\frac{1}{2}}$, $s_1 = \frac{1}{16}r_1^2 + \frac{1}{16}r_2^2 + \gamma^5$, $s_2 = \frac{1}{8}r_1 r_2$. For example, let

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0. \tag{25}$$

Here $\gamma = 1$, $\delta = -8$, $\varepsilon = 29$, $\zeta = -480$, $r_1 = 480$, $r_2 = -476$, $s_1 = 28562$, $s_2 = -28560$, and these may be substituted at once in (7). When $\gamma^3 - \gamma\varepsilon + \delta^2 = 0$, the expression for r_2 must be replaced by $-(\gamma^{-2}\delta^3 + 4\gamma\delta)$. Thus, if

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0, \tag{26}$$

we have $\gamma = 2$, $\delta = 2$, $\varepsilon = 6$, $\zeta = 10$, $r_1 = -10$, $r_2 = -18$, $s_1 = \frac{117}{2}$, $s_2 = \frac{45}{2}$, and by (7) we have $u_1^5 = 2$, $u_2^5 = -4$, $u_3^5 = 8$, $u_4^5 = -16$, whence $x_1 = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}}$.

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[*] "Resolution of Solvable Equations of the Fifth Degree," *American Journal of Mathematics*, VI, 103 (1883).

[†] Vol. VI, p. 313. [‡] Vol. VI, p. 311.

9. Thus far we have had only a statement of certain results without proofs, brought together for the convenience of those who may have occasion to deal practically with the problem of the resolution of the quintic. Let us now consider the subject in due order.

10. Referring to the general equation (1), let $y = x + ba^{-1}$; then substituting,

$$y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon x + \zeta = 0, \tag{27}$$

where the coefficients have the values shown in (2). Let

$$x_1 + \omega^m x_2 + \omega^{2m} x_3 + \omega^{3m} x_4 + \omega^{4m} x_5 = 5u_m, \tag{28}$$

where m has any integral value from 1 to 4 inclusive. From this, bearing in mind that $x_1 + x_2 + x_3 + x_4 + x_5 = -5ba^{-1}$, and that $\omega^m + \omega^{2m} + \omega^{3m} + \omega^{4m} = -1$, we have at once the proof of (6). We may write y for x in (28) without changing the result, since the addition of ba^{-1} to each root in the first member adds merely to the second member the expression $ba^{-1}(1 + \omega^m + \omega^{2m} + \omega^{3m} + \omega^{4m}) = 0$. Then, on development,

$$\begin{aligned}
\Sigma y^2 &= 10\,(u_1u_4 + u_2u_3), \\
\Sigma y^3 &= 15\,(u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4), \\
\Sigma y^4 &= 30\,(u_1^2u_4^2 + u_2^2u_3^2) + 20\,(u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1) + 120u_1u_2u_3u_4, \\
\Sigma y^5 &= 5\Sigma u^5 + 100\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_2^3u_1u_3 + u_3^3u_4u_2) \\
&\qquad + 150\,(u_1^2u_2^2u_4 + u_4^2u_3^2u_1 + u_2^2u_4^2u_3 + u_3^2u_1^2u_2).
\end{aligned}$$

But $\Sigma y^2 = -20\gamma$, $\Sigma y^3 = -30\delta$, $\Sigma y^4 = 200\gamma^2 - 20\varepsilon$, $\Sigma y^5 = 500\gamma\delta - 5\zeta$. Hence, by comparison with the above, we obtain four equations, which contain as unknown quantities the four elements, and which we may refer to hereafter as the Eulerian equations,* namely,

$$-2\gamma = u_1u_4 + u_2u_3, \tag{29}$$

$$-2\delta = u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4, \tag{30}$$

$$-\varepsilon = u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1 + 3u_1u_2u_3u_4 - 4\gamma^2, \tag{31}$$

$$-\zeta = \Sigma u^5 - 10\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_3^2u_1u_3 + u_3^3u_4u_2) + 20\gamma\delta. \tag{32}$$

11. The elements (u_1, u_2, u_3, u_4) were discovered and discussed independently by Euler and Bezout,† and further light was thrown on the theory by Lagrange and Vandermonde, also independently.‡ Such of their results as we shall need

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* This succinct exposition of the theory as it stood in 1771 is substantially copied from Mr. Harley's "Theory of Quintics," Quarterly Journal of Mathematics, January, 1860.

† Euler, "De Resolutione Æquationum cuiusvis Gradus," Novi Comm. Petrop. IX (1762–63); Bezout, "Sur la Résolution générale des Équations de tous les Degrés," Paris Mémoires, 1765.

‡ The best known paper of Lagrange on algebraic equations and Vandermonde's paper on the resolution of equations were read about the same time, in 1771, before the academies of Berlin and Paris respectively.

to make use of are embodied in the preceding paragraph. Lagrange was the first* to recommend the resolution of the quintic by the use of an auxiliary sextic. He showed that Σu^5, or $u_1^5 + u_2^5 + u_3^5 + u_4^5$, has six values, which he proposed to take as the roots of a resolvent sextic, whose coefficients might be determined in terms of those of the quintic. Supposing that by such a resolvent there were once ascertained a value of Σu^5, say $-p_3$, he showed that a biquadratic might be taken,

$$w^4 + p_3 w^3 + p_2 w^2 + p_1 w + p_0 = 0, \tag{33}$$

of which the four roots would be u_1^5, u_2^5, u_3^5, and u_4^5,† and of which the remaining coefficients, p_2, p_1, p_0, might be determined by a process which would have been mathematically feasible, though in practice involving much labor, and which was never carried out in detail by any one, so far as appears. It is but a momentary digression from our course to inquire into the real form of the unnecessary biquadratic in Lagrange's theory, as a mere matter of historic interest. The four values of w are given in (7). Putting these together to form (33), we see that $p_3 = -r_1$, $p_2 = \frac{1}{4}r_1^2 - \frac{1}{4}r_2^2 - 2r_3$, $p_1 = r_1 r_3 + r_2 r_4$, $p_0 = r_3^2 - r_4^2$, where r_1 and r_2 have the same values as in (7), and $r_3 = \gamma^5 + 10\gamma^3 v + 5\gamma v^2$, $r_4 = (5\gamma^4 + 10\gamma^2 v + v^2)v^{\frac{1}{2}}$. Since p_0 is the product of the roots of (33), it is the fifth power of the "resolvent product" $u_1 u_2 u_3 u_4$, and Lagrange's method of resolution was so far modified by Messrs. Cockle, Harley, and Cayley‡ as to start with p_0 as the first coefficient of the biquadratic to be determined, for employment instead of p_3 as the basis for finding the others; a proceeding equally laborious with that suggested by Lagrange, and apparently never undertaken. A slight improvement upon Lagrange's proposal for a resolvent was suggested by Meyer Hirsch in 1808, without practical result.§ We may illustrate (33) by the first example (12) before chosen, wherein $v = \frac{1}{20}$, $\gamma = -\frac{7}{2}$, $r_1 = 90$, $r_2 = 6\sqrt{5}$, so that $r_3 = -\frac{5467}{10}$, $r_4 = \frac{18911}{250}\sqrt{5}$, and the biquadratic is

$$w^4 - 90w^3 + \frac{15367}{5}w^2 - \frac{1173342}{25}y + \frac{844596301}{3125} = 0. \tag{34}$$

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* For a historical sketch of the views of different writers on the solvability of the quintic, see Cockle, "On the Method of Symmetric Products," Philosophical Magazine, Feb. 1854. The impossibility of a solution was not conceded by all—Jerrard being apparently the last disputant—before 1863; see Cockle, "Concluding Remarks," Philosophical Magazine, Sept., 1863. Schulenburg's "solution" was translated and printed in the Analyst in 1877. Meyer Hirsch's "solution" was once famous.

† The elements discussed by Lagrange were five times as large as these, with corresponding differences throughout.

‡ Cayley, "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," Philosophical Transactions, Vol. 151, p. 263 (1861). This paper contained, as will be stated later, an unexceptionable method for determining p_0.

§ Ross, Translation of Hirsch's Sammlung, p. 282.

12. Lagrange afterwards reviewed his theory of equations in comparison with the closely similar theory of Vandermonde. Of this review we need to note but a single point.* Any one who takes the trouble to develop u_1^5, u_2^5, u_3^5, u_4^5, as the elements are defined in (28), will find that $u_1^5 + u_2^5 + u_3^5 + u_4^5$, say r_1, is a rational function of the roots, and that $u_1^5 + u_4^5 - u_2^5 - u_3^5$, say r_2, is another rational function multiplied by $\sqrt{5}$. In carrying out a certain numerical example, Lagrange found the fifth-powers of the elements, according to his theory, to be of a form which we may write briefly thus:

$$\left.\begin{aligned} u_1^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + q_1, & u_2^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + q_2, \\ u_4^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - q_1, & u_3^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - q_2. \end{aligned}\right\} \qquad (35)$$

This arrangement is indeed a necessary one, if we define q_1 to be $\frac{1}{2}(u_1^5 - u_4^5)$, and q_2 to be $\frac{1}{2}(u_2^5 - u_3^5)$. But Vandermonde had worked out the same example thus:

$$\begin{aligned} u_1^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + q_1, & u_2^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - q_1, \\ u_4^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - q_2, & u_3^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + q_2. \end{aligned}$$

In comparing Vandermonde's result with that obtained by his own method, Lagrange pointed out that the former was erroneous even according to Vandermonde's theory, as he showed by revising Vandermonde's work and producing the correct result (35).

13. It was not until 1858 that any one noticed that Gian Francisco Malfatti† had, in the very same year that Lagrange and Vandermonde brought out their almost identical theories, done more than either of them towards the resolution of quintic equations.‡ For eighty seven years his successful method of resolution was looked upon merely as a fruitless effort towards algebraic solution, as indeed Malfatti himself would seem to have regarded it. He pursued a method for

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* Lagrange, Théorie des Équations Numériques, Note XIV, paragraphs 30–36.

† My authorities concerning Malfatti are Harley, "On Recent Researches in the Theory of Equations," Manchester Proceedings, Nov. 3, 1863 (who quotes from Brioschi, "Sulla Risolvente di Malfatti per le Equationi del Quinto Grado," Reale Istituto Lombardo—Rendiconti, IX, 215); Brioschi, "Ueber die Auflösung der Gleichungen vom fünften Grade," Mathematische Annalen, XIII, 109; and Minich, "Sulle Equazioni del Quinto Grado" (I have seen only the second chapter of the "first memoir"), Atti del Reale Istituto Veneto (5th series), VIII, 893. Mr. Harley's paper contains a striking statement of the enormous difficulties attending the solution of the Eulerian equations by any usual method of elimination.

‡ Like most sweeping statements, this remark, that no one noticed what Malfatti had done, has its exception. Ruffini wrote in 1805 (Memorie della Società Italiana, XII, 322): "L'illustre Autore ha così il vanto di avere, senza punto conoscere le corrispondenti osservazioni del Lagrange, ottenuta completamente una trasformazione, rapporto alla quale l'immortale Matematico di Torino non aveva infine determinato che il grado, ed un solo coefficiente." The last four words might well have been omitted. Malfatti's memoir, "De Æquationibus Quadrato-cubicis Disquisitio Analytica," appeared in the Siena transactions for 1771.

solving equations of the lower degrees by means of a resolvent, the resolvent in each case being one degree lower than the equation to be solved. In applying this method to the quintic, he actually obtained, to his disappointment, a resolvent sextic, in which the unknown quantity was what I should denote by $25v + 5\gamma^2 + \frac{5}{3}\varepsilon$. I presume that he did this by the solution of the Eulerian equations, or by some equivalent process, and it is certain that he indicated some simpler method for the final determination of the elements than the solution of a biquadratic. But with all this he had a sense of failure, for he continued during many years to avow himself a believer in the solvability of the quintic. Possessed of high reputation in Italy, he was chiefly known in other countries* as taking a conspicuous part on the wrong side of that question. The result even in Italy was to deter inquirers from study of his writings on the quintic, which are indeed said to have been superciliously disparaged and before long overlooked and forgotten. The existence of Malfatti's resolvent was brought to the notice of the Venetian Institute, in 1858, by Professor Minich, and a paper concerning it was read in 1863 by Signor Brioschi at Milan.

14. Meantime, all went on as if Malfatti's work had never been done. A resolvent was wanted. In 1835 Jacobi† showed that a resolvent can be constructed having for its roots the six values of a quantity which we may, following Professor Cayley, denote by ϕ, equal in our notation to $10\sqrt{(5v)}$. His reasoning would seem to have been substantially as follows, or at any rate the following reasoning will justify his conclusions.

15. Let C represent a cycle, by which term let us understand the sum of five similar functions of the roots $(x_1, x_2, x_3, x_4, x_5)$, in which the subscript numbers differ cyclically according to a fixed sequence. If, for example, the sequence is 12345, we may have

$$\mathrm{C}x_1^2x_4 = x_1^2x_4 + x_2^2x_5 + x_3^2x_1 + x_4^2x_2 + x_5^2x_3,$$

while if the sequence is 12354,

$$\mathrm{C}x_1^2x_4 = x_1^2x_4 + x_2^2x_1 + x_3^2x_2 + x_5^2x_3 + x_4^2x_5.$$

When necessary to distinguish different sequences, let 12345, 13425, 14235, 12534, 12453, 13254, be denoted respectively by the symbols C_1, C_2, C_3, C_4, C_5, C_6, and 13524, 12354, 12543, 13245, 14325, 12435, by C'_1, C'_2, C'_3, C'_4, C'_5, C'_6. (I have here followed Mr. Cayley's order.) These twelve comprise all possible cycles, since reversing the sequence—as 15432 for 12345—does not change the cycle. We may look on $\mathrm{c}_1 x_1 x_2$ as a group of five products, each of two different roots, in which group each root appears twice. Conversely, every such group is a cycle comprising the products of adjacent roots, and the number of possible cycles (12) is the number of possible groups. Similarly, $\mathrm{c}_1 x_1 x_3$ is a group of the same sort. Conversely, every such group is a cycle comprising the products of non-adjacent roots, the possible number being 12. In fact, $\mathrm{c}_1 x_1 x_2 = \mathrm{c}'_1 x_1 x_2$, and so on, the roots being adjacent in the former cycle and non-adjacent in the latter. Let $\phi_1 = \mathrm{c}_1 x_1 (x_2 - x_3)$, a cycle comprising the difference between the products of adjacent and non-adjacent roots. It will be found that $\mathrm{c}'_1 x_1 (x_3 - x_5) = -\phi_1$. Taking all possible combinations, we have

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* Peacock, Report to the British Association, 1833, p. 312.

† Jacobi, "Observatio de Æquatione Sexti Gradus ad quam Æquationes Quinti Gradus Revocari Possunt," in a paper entitled "Observatiunculæ ad Theoriam Æquationum Pertinentes," Crelle, XIII, 340, as cited by Cayley, "On Tschirnhausen's Transformation," Philosophical Transactions, v. 152, p. 578.

$$\left.\begin{array}{ll} \phi_1 = \mathrm{c}_1 x_1 (x_2 - x_3) = -\mathrm{c}'_1 x_1 (x_3 - x_5), & \phi_4 = \mathrm{c}_4 x_1 (x_2 - x_5) = -\mathrm{c}'_4 x_1 (x_3 - x_2), \\ \phi_2 = \mathrm{c}_2 x_1 (x_3 - x_4) = -\mathrm{c}'_2 x_1 (x_2 - x_3), & \phi_5 = \mathrm{c}_5 x_1 (x_2 - x_4) = -\mathrm{c}'_5 x_1 (x_4 - x_3), \\ \phi_3 = \mathrm{c}_3 x_1 (x_4 - x_2) = -\mathrm{c}'_3 x_1 (x_2 - x_5), & \phi_6 = \mathrm{c}_6 x_1 (x_3 - x_2) = -\mathrm{c}'_6 x_1 (x_2 - x_4). \end{array}\right\} \quad (36)$$

The function ϕ has therefore 12 values, of which six are respectively identical with the other six, but with contrary signs. Hence ϕ^2 is a six-valued function, and if its values be taken as the roots of a sextic, the coefficients of the sextic will be rational and integral symmetric functions of the roots of the quintic, and therefore rational and integral functions of the coefficients of the quintic. Regarding this sextic as the product of a sextic whose roots are the values of ϕ by another whose roots are the values of $-\phi$, we have for ϕ a sextic whose coefficients can contain only rational and integral functions of the coefficients of the quintic or square roots of such functions. Suppose that two roots are equal, say $x_5 = x_4$, and that the discriminant ($\square$) therefore vanishes, by the usual theory. Then $\mathrm{c}_1 = \mathrm{c}'_2$, so that $\phi_1 = -\phi_2$, and similarly $\phi_3 = -\phi_5$ and $\phi_4 = -\phi_6$. Hence ϕ^2 has in this case but three values, the roots of a cubic, so that the odd powers of the sextic severally vanish, whence we must infer that some power of the discriminant is a factor of each coefficient of an odd power in the sextic for ϕ. In this sextic, since ϕ is a function of the roots of two dimensions, the coefficient of ϕ must be a function of 10 dimensions, that of ϕ^3 of 6 dimensions, and that of ϕ^5 of 2 dimensions, while $\sqrt{\square}$ has ten dimensions, and this can appear only in the coefficient of ϕ, whence it follows that

$$\phi^6 + a_2\phi^4 + a_4\phi^2 \pm n\sqrt{\square}\,\phi + a_6 = 0. \qquad (37)$$

This is the form given by Jacobi, n being a numerical factor whose value he determined correctly, leaving the other coefficients to be investigated by others.

16. Jacobi's resolvent is substantially the same as Malfatti's, the one being the result of an endeavor to improve on Lagrange's proposal by building up a sextic from a chosen function of the roots, the other the result of elimination from the Eulerian equations or their equivalent. The identity of the two resolvents is shown by an equation which, as v has not yet been formally defined, we shall adopt as a definition, namely,

$$v = \frac{1}{500}\phi^2. \tag{38}$$

The following might, however, be taken as the definition of the same symbol:

$$v = \gamma^2 - u_1u_2u_3u_4. \tag{39}$$

Or, we might define $v^{\frac{1}{2}}$ by a separate symbol, as has already been done by at least one writer.* It will then be easy to define v as the square of $v^{\frac{1}{2}}$. Thus, observing (29), we may put

$$v^{\frac{1}{2}} = \gamma + u_1u_4 = -\gamma - u_2u_3. \tag{40}$$

The connection between (39) and (40) is obvious, since (39) may be derived from (40) by multiplying the value of u_1u_4 by that of u_2u_3. The connection between (38) and (40) becomes clear when we substitute for the elements their values as defined by (28), aided by one or two of the most elementary formulæ in the theory of symmetric functions. We find, for instance, that

$$25u_1u_4 = \mathrm{c}_1\left(x_1^2 + \omega x_1x_2 + \omega^2x_1x_3 + \omega^3x_1x_3 + \omega^4x_1x_2\right) = -25\gamma \pm \tfrac{1}{2}\sqrt{5}\phi.$$

17. The use of a special symbol for the quantity v may not seem to constitute an important step in advance in the theory of this well-known resolvent. It is to be observed, however, that the relations of this quantity with t, as shown in (5), make the introduction and discussion of it a necessity in any event; that either $v^{\frac{1}{2}}$ or some other symbol for the same quantity is needed in (40) and elsewhere, and that $v^{\frac{1}{2}}$ is to be preferred as marking the fact that it is an irrational function of the roots; and that the use of the quantity ϕ instead of v is not indicated, after the rudiments are left behind, by any analytic advantage. It will be observed that v is rational whenever t is, and *vice versa*; that v is rational whenever ϕ is, and sometimes when ϕ is not. Finally, we shall see that the use of a symbol for the quantity v facilitates materially the derivation of the three chief resolvents from the Eulerian equations.†

18. Unaware of what had been done by Malfatti and Jacobi, three English analysts, by one step after another, went on improving the theory as then known

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* Schläfli, in a paper referred to later.

† I observe that in one place Signor Brioschi employs a symbol for v, but only momentarily, as a stepping stone from $\gamma^2 - v$ to $v^{\frac{1}{2}}$. See his paper previously cited.

until the same resolvent, in Jacobi's form, was brought out in the fullest detail. Mr. (now Sir James) Cockle discovered a resolvent for the special case $x^5 + 10dx^2 + f = 0$, identical, as far as it goes, with that of Malfatti; Mr. Harley worked up the subject further, and Mr. Cayley, carrying out elaborately a hint given by Mr. Harley, calculated the complete resolvent.* About the same time Captain von der Schulenburg gave the same resolvent in the course of what he supposed to be a solution of the quintic.† He made no claim to novelty, and may be presumed to have seen Jacobi's paper. Finally, the same form of Malfatti's resolvent was rediscovered about 1863 by Professor Minich,‡ who was then unaware of what had been done in Germany and England.

19. Some account of the reinstatement of Malfatti's resolvent has already been given. In its presentation by Signor Brioschi, the work was simplified by suppressing δ and afterwards showing that the result was not thereby affected. For solving the Eulerian equations two auxiliary quantities (besides seven others afterwards eliminated) were introduced, namely, r, which I call $-\delta - tv^{\frac{1}{2}}$, and v, which I call $\gamma^2 - v$. Both were finally eliminated to produce the (dexter) resolvent in x, which I call $v^{\frac{1}{2}}$, substantially identical with that of Jacobi. Whether these details were taken from Malfatti or introduced by Signor Brioschi, I do not know, but infer that several simplifications were made by the latter. The relations given by the Eulerian theory were made use of to determine the values of u_1u_4, u_2u_3, $u_1^2u_3$, etc., and the values of the elements were obtained by the formulæ

$$\left.\begin{array}{ll} u_1^5 = (u_1^2u_3)^2\,u_2^2u_1\,(u_2u_3)^{-2}, & u_3^5 = (u_3^2u_4)^2\,u_1^2u_3\,(u_1u_4)^{-2}, \\ u_2^5 = (u_2^2u_1)^2\,u_4^2u_2\,(u_1u_4)^{-2}, & u_4^5 = (u_4^2u_2)^2\,u_3^2u_4\,(u_2u_3)^{-2}. \end{array}\right\}. \qquad (41)$$

A somewhat different statement of Malfatti's method was published in 1869 by Professor Schläfli.§ The two quantities taken by him for discussion, after reducing the Eulerian equations to manageable form in substantially the same manner as that exhibited by Signor Brioschi, are t, which I call $v^{\frac{1}{2}}$, and another which I call $s^{\frac{1}{2}}$, and which—see (30)—may be explained thus:

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* Cockle, "Researches in the Higher Algebra," Manchester Memoirs, XV (1858), 131; "Theory of Equations of the Fifth Degree," Philosophical Magazine, July, 1859; Harley, "On the Method of Symmetric Products," Manchester Memoirs, XV (1859); "On the Theory of Quintics," Quarterly Journal, Jan., 1860; Cayley, "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," Philosophical Transactions, 151 (1861), p. 263. Mr. Cayley's statement (1862) of Jacobi's prior discovery has already been cited.

† Already referred to: Auflösung der Gleichungen Fünften Grades, Halle, 1861. Mr. Cayley's memoir had been read in February of the same year.

‡ See his statement, Atti del Reale Istituto Veneto, VIII, 906-908 (1882).

§ "La Risolvente dell' Equazione di Quinto Grado sotto la Forma di un Determinante Simmetrico a Quattro Linee," Annali di Matematica, 2d Series, III, 171.

$$s^{\frac{1}{2}} = \delta + u_2^2u_1 + u_3^2u_4 = -\delta - u_1^2u_3 - u_4^2u_2. \tag{42}$$

He finally obtains a single equation of the twelfth degree in t, which I call $v^{\frac{1}{2}}$, in the form of a symmetric determinant. Of this the first row and first column contain only odd powers of the variable, and the other constituents only even powers, so that the determinant becomes, on evaluation, a sextic in t^2, equivalent in substance to Malfatti's resolvent. It is only another instance of the simplification resulting everywhere from the use of a symbol for the quantity v that, if we change the notation and divide one row and one column by $v^{\frac{1}{2}}$, Professor Schläfli's determinant becomes expressly a sextic, namely,

$$\begin{vmatrix} \mathrm{A}, & \mathrm{H}, & \mathrm{G}, & \zeta \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F}, & \mathrm{L} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C}, & \delta \\ \zeta, & \mathrm{L}, & \delta, & -\gamma \end{vmatrix} v = 0, \tag{43}$$

where

$$\begin{aligned}
\mathrm{A} &= 625v^2 - 25(20\gamma^2 + 7\varepsilon)v + 100\gamma^4 + 45\gamma^2\varepsilon + 31\varepsilon^2 + 25\gamma\delta^2 - 16\delta\zeta + (25\gamma^6 - 35\gamma^4\varepsilon \\
&\qquad + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon + 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3)v^{-1}, \\
\mathrm{B} &= -250\gamma v + 25\gamma^3 - 10\gamma\varepsilon + 16\delta^2, \\
\mathrm{C} &= 25v + \gamma^2 - \varepsilon, \\
\mathrm{F} &= -\gamma\delta + \zeta, \\
\mathrm{G} &= -25\gamma v + 35\gamma^3 - 19\gamma\varepsilon + 25\delta^2, \\
\mathrm{H} &= 25\delta v + 15\gamma^2\delta + 19\delta\varepsilon - 24\gamma\zeta, \\
\mathrm{L} &= 25v - \varepsilon.
\end{aligned}$$

That he did not himself put it in this shape can only be due, I think, to his being embarrassed by the second irrational, $s^{\frac{1}{2}}$, the value of which he proceeds to determine, after finding $v^{\frac{1}{2}}$, by means of the minors of his determinant. (We shall see further on how, after introducing the quantity t, which, like v and s, is a rational function of the roots, the same minors, in simpler form, may be used, if any one chooses that course, to better purpose.) Having ascertained $v^{\frac{1}{2}}$ and $s^{\frac{1}{2}}$, he employs, like Signor Brioschi, the relations given in (29), (30), (40), and (42) to determine the values of u_1u_4, u_2u_3, $u_1^2u_3$, $u_2^2u_1$, $u_4^2u_2$, and thence he derives those of the elements by these formulæ: $u_1^5 = (u_1^2u_3)^2u_2^2u_1(u_2u_3)^{-2}$, $u_2 = u_1^2 . u_4^2u_2(u_1u_4)^{-2}$, $u_3 = u_1^2u_3 . u_1^{-2}$, $u_4 = u_1u_4 . u_1^{-1}$.

20. Two lines of progress will have been noticed in the historical sketch just given. The building up of a sextic upon a simple function of the roots was

suggested by Lagrange and accomplished finally by Mr. Cayley, but this did not result in expressing the elements in terms of that function.* On the other hand, Malfatti and his followers obtained substantially the same sextic by elimination, without regard to the relation of the variable to the roots, but with the advantage of determining the elements as the immediate result of a solution of the sextic. It will presently be found that the function of the roots, ϕ, upon which Jacobi's form of Malfatti's resolvent is built up, is only one, and not in all respects the most important, of three such functions, and we shall accordingly proceed to discuss the two other resolvents. Examining the theory deduced by elimination, we shall find that the introduction of a new quantity, t, and of a special symbol for a quantity hitherto neglected, v, lead to notable simplifications in the derivation and form of the two fundamental equations referred to in the preceding paragraph, and that from these equations each of the three resolvents can at once be derived. We shall find it possible also to deduce simple expressions by which to determine the elements in terms of t and v, or t and s, and the coefficients of the quintic.

21. Any six-valued function of the roots may be used for a resolvent sextic, including, in certain cases, six values of a twelve-valued function. Of such functions the number is unlimited, because the simpler six-valued functions admit of any number of six-valued combinations. Comparatively few six-valued functions have been definitely pointed out. That first suggested by Lagrange, Σu^5, is complex. A slightly simpler function for the same purpose was suggested by Hirsch, viz., $2\mathrm{c}_1 x_1^3 (x_2 x_5 + x_3 x_4) + 3\mathrm{c}_1 x_1^2 x_2 (x_3^2 + x_2 x_4)$. But it may be remarked incidentally that the only sextic needed for the immediate determination of the values of Σu^5 is one in which the roots are of the form $\mathrm{c}_1 y_1^2 y_2 (y_3^2 + y_2 y_4)$, where $y = x + ba^{-1}$, as in (27). The function $\mathrm{c}_1 x_1 (x_2 - x_3) x_5$ was observed by Schulenburg as having six values. M. Hermite suggested, and Dr. Salmon† carried out, the calculation of a sextic whose roots are squares of functions of the form $\mathrm{c}_1 x_1^2 (x_2^2 x_3 - x_3^2 x_5 + x_4^2 x_2 - x_5^2 x_4 + 2x_3 x_4 x_5 - 2x_2 x_3 x_4)$, or $(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_5 - x_1)$. This function does not seem well suited to serve as a resolvent. Of all six-valued functions of the roots of the quintic, I find three pre-eminently suitable for the formation of resolvent sextics. These are $\mathrm{c}_1 x_1 (x_2 - x_3)$, which we call ϕ after Mr. Cayley; $\mathrm{c}_1 x_1 (x_2 - x_3) x_5$, which let us call ψ; and $\phi\psi^{-1}$,

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* Attention was called to this defect in a paper by Cockle, "On the Theory of Equations," Manchester Proceedings, Nov., 1863, p. 171.

† Modern Higher Algebra, 3d Ed., pp. 228–230.

which let us call τ. The separate values of ϕ have been shown in (36). Those of τ are $\tau_1 = \phi_1\psi^{-1}$, $\tau_2 = \phi_2\psi^{-2}$, and so on, and those of ψ are

$$\left.\begin{array}{ll} \psi_1 = \mathrm{c}_1 x_1 (x_2 - x_3) x_5 = -\mathrm{c}_1' x_1 (x_3 - x_5) x_4, & \psi_4 = \mathrm{c}_4 x_1 (x_2 - x_5) x_4 = -\mathrm{c}_4' x_1 (x_3 - x_2) x_5, \\ \psi_2 = \mathrm{c}_2 x_1 (x_3 - x_4) x_5 = -\mathrm{c}_2' x_1 (x_2 - x_3) x_4, & \psi_5 = \mathrm{c}_5 x_1 (x_2 - x_4) x_3 = -\mathrm{c}_5' x_1 (x_4 - x_3) x_5, \\ \psi_3 = \mathrm{c}_3 x_1 (x_4 - x_2) x_5 = -\mathrm{c}_3' x_1 (x_2 - x_5) x_3, & \psi_6 = \mathrm{c}_6 x_1 (x_3 - x_2) x_4 = -\mathrm{c}_6' x_1 (x_2 - x_4) x_5. \end{array}\right\} \quad (44)$$

22. Of the three functions just mentioned, only ϕ has, I think, been suggested heretofore as suitable for a resolvent. It underlies that resolvent whose history, under various forms, has been given above, the only resolvent hitherto found feasible.* In his investigation of the sextic in ϕ, Mr. Cayley proved that function to be a seminvariant, or function which remains unaltered when any given quantity is added to each root. (This fact is made instantly visible by the use of our notation, since $\mathrm{c}_1 [x_1 + k][x_2 + k - x_3 - k] = \mathrm{c}_1 x_1 [x_2 - x_3] + k\mathrm{c}_1 [x_2 - x_3]$, the last term vanishing.) The coefficients, he remarked, must therefore all be seminvariants, as follows from their being functions of the several values of ϕ. He then shortened the calculation of them by assuming that one root vanished, whence $f = 0$, and afterwards filling out the missing terms by reference to the known properties of seminvariants. The full values of the coefficients as computed by him will be given presently. The independent term, equal to $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, is stated by him at length, and I find it reducible to this shorter expression:

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 = 40000a^{-6}(al_0' - 25c_0'^2). \qquad (45)$$

Here l_0' is the leading coefficient of the simplest linear covariant of the quintic, say $l_0'x + l_1'$, the full expression for which will be stated further on, as given in various standard works; and $\mathrm{c}_0' = ace - ad^2 - b^2e + 2bcd - c^3$, the leading coefficient of the canonizant. Or,

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 = 40000(l_0 - 25c_0^2), \qquad (46)$$

where l_0 and c_0 have the same values as in paragraph 2. The coefficients of the sextic in ϕ, which for distinction I will refer to as the dexter resolvent, are best exhibited, I think, in the following form:

$$\left.\begin{array}{c} \phi^6 + 100(-3\gamma^2 - \varepsilon)\phi^4 + 2000(15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)\phi^2 \\ \pm 800\sqrt{(5\mathrm{D})} + 40000(l_0 - 25c_0^2) = 0. \end{array}\right\} \quad (47)$$

Here D, the discriminant × a^{-8}, has the same value as in (19). For brevity, let us express (47) thus:

$$\phi^6 + 100d_2\phi^4 + 2000d_4\phi^2 \pm 800\sqrt{(5\mathrm{D})}\,\phi + 40000d_6 = 0. \qquad (48)$$

---

*I exclude, of course, sextics called resolvents in the solution of the quintic by elliptic functions.

Since by (38) we put $v = \frac{1}{500}\phi^2$, we have $\phi = 2\sqrt{5}\,(5v^{\frac{1}{2}})$, and (48) becomes, after dividing by 40000,

$$\tfrac{1}{5}(5v^{\frac{1}{2}})^6 + d_2(5v^{\frac{1}{2}})^4 + d_4(5v^{\frac{1}{2}})^2 \pm \tfrac{1}{5}\sqrt{\mathrm{D}}\,(5v^{\frac{1}{2}}) + d_6 = 0. \tag{49}$$

Transposing the odd power, and squaring,

$$\left.\begin{array}{l} 5^{10}v^6 + 2.5^9d_2v^5 + (2.5^7d_4 + 5^8d_2^2)\,v^4 + 2\,(5^5d_6 + 5^6d_2d_4)\,v^3 \\ \quad + (2.5^4d_2d_6 + 5^4d_4^2)\,v^2 + (2.5^2d_4d_6 - \mathrm{D})\,v + d_6^2 = 0. \end{array}\right\} \tag{50}$$

A very slight modification of this will give the sextic of Malfatti, mentioned in paragraph 13. It will be seen that (50) coincides with the determinant (43), and we may remark that

$$\begin{vmatrix} 25\gamma^3 - 10\gamma\varepsilon + 16\delta^2, & \gamma\delta - \zeta, & \varepsilon \\ \gamma\delta - \zeta, & \gamma^2 - \varepsilon, & \delta \\ \varepsilon, & \delta, & -\gamma \end{vmatrix} = d_6. \tag{51}$$

For the present, we may consider the coefficients of the dexter resolvent, so far as they were left undetermined by Jacobi (see paragraph 15), as having been proved for us by the process followed by Mr. Cayley, of which it is not necessary to repeat the details. A different proof will be given when we discuss the Eulerian equations. When $v = \gamma^2$, (50) becomes (19). When $v = 0$, $d_6 = 0$, which proves (24).

23. For the second function named at the close of paragraph 21 we are now enabled to obtain at once what I call the sinister resolvent, which is merely the dexter resolvent of the quintic when the order of the coefficients is reversed, f being interchanged with a, e with b, and so on. This is equivalent to supposing that x is replaced by x^{-1}, when ϕ_1, or $\mathrm{C}_1x_1(x_2 - x_3)$, is replaced by $\mathrm{C}_1x_1^{-1}(x_2^{-1} - x_3^{-1})$, which is equal to $-af^{-1}\psi_1$, since $af^{-1} = -x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$ and $\psi_1 = \mathrm{C}_1x_4(x_3 - x_2)\,x_5$, as will be seen on comparison. The other values of ϕ are similarly equal to the corresponding values of $-af^{-1}\psi$. The dexter resolvent of the reversed quintic is therefore a sextic in ψ, and we may write it out by reversing the several letters as stated. Mr. Cayley's determination of the *dexter resolvent* is as follows, except that I abbreviate the last term as already explained:

$$\left.\begin{array}{l} a^6\phi^6 - 100a^4(ae - 4bd + 3c^2)\,\phi^4 + 2000a^2\,(-2a^2df + 3a^2e^2 + 6abcf \\ \quad - 14abde - 2ac^2e + 8acd^2 - 4b^3f + 10b^2ce + 20b^2d^2 - 40bc^2d \\ \quad + 15c^4)\,\phi^2 \pm 800a^2\sqrt{(5\square)}\,\phi + 40000\,(al_0' - 25c_0'^2) = 0. \end{array}\right\} \tag{52}$$

Here $\square$ = discriminant, and l_0' and c_0' have the same meaning as in the preceding paragraph, namely,

$$
\begin{aligned}
l_0' = {} & a^2cf^2 - 2a^2def + a^2e^3 - ab^2f^2 - 4abcef + 8abd^2f - 2abde^2 - 2ac^2df + 14ac^2e^2 \\
& - 22acd^2e^* + 9ad^4 + 6b^3ef - 12b^2cdf - 15b^2ce^2 + 10b^2d^2e + 6bc^3f + 30bc^2de \\
& - 20bcd^3 - 15c^4e + 10c^3d^2, \\
c_0' = {} & ace - ad^2 - b^2e + 2bcd - c^3.
\end{aligned}
$$

These are respectively the leading coefficients of the simplest linear and cubic covariants of the quintic, the other coefficients being

$$
\begin{aligned}
l_1' = {} & a^2df^2 - a^2e^2f - 2abcf^2 - 4abdef + 6abe^3 + 8ac^2ef - 2acd^2f - 12acde^2 + 6ad^3e \\
& + b^3f^2 - 2b^2cef + 14b^2d^2f - 15b^2de^2 - 22bc^2df + 10bc^2e^2 + 30bcd^2e \\
& - 15bd^4 + 9c^4f - 20c^3de + 10c^2d^3, \\
c_1' = {} & acf - ade - b^2f + bce + bd^2 - c^2d, \\
c_2' = {} & adf - ae^2 - bcf + bde + c^2e - cd^2, \\
c_3' = {} & bdf - be^2 - c^2f + 2cde - d^3.
\end{aligned}
$$

Reversing the letters in (52) as just explained, we have the *sinister resolvent*,

$$
\left.
\begin{aligned}
& a^6\psi^6 - 100a^4(bf - 4ce + 3d^2)\psi^4 + 2000a^2(-2acf^2 + 6adef - 4ae^3 \\
& + 3b^2f^2 - 14bcef - 2bd^2f + 10bde^2 + 8c^2df + 20c^2e^2 - 40cd^2e \\
& + 15d^4)\psi^2 \mp 800af\sqrt{(5\square)}\,\psi + 40000(fl_1' - 25c_3'^2) = 0.
\end{aligned}
\right\} \qquad (53)
$$

24. The elements were shown in paragraph 10 to be seminvariants, so that in using any resolvent we shall find it well to employ functions of the roots that are themselves seminvariants. This was done necessarily in paragraph 22, in discussing the dexter resolvent, since ϕ and v are seminvariants. It is better to modify the sinister resolvent somewhat, since ψ is not a seminvariant.† Any function of the roots can be transmuted into a seminvariant by deducting from each root one-fifth of the sum of all, that is, by adding ba^{-1} to each root. In lieu of ψ, equal to $\mathrm{c}_1x_1(x_2 - x_3)x_5$, let us take σ, denoting $\mathrm{c}_1(x_1 + ba^{-1})(x_2 - x_3)(x_5 + ba^{-1})$. This is the same as $\mathrm{c}_1x_1(x_2 - x_3)(x_5 + ba^{-1}) + \mathrm{c}_1ba^{-1}(x_2 - x_3)(x_5 + ba^{-1})$. The latter term vanishes, since $\mathrm{c}_1ba^{-1}x_2x_5 - \mathrm{c}_1ba^{-1}x_5x_3 = 0$ and $\mathrm{c}_1b^2a^{-2}(x_2 - x_3) = 0$. Hence $\sigma = \mathrm{c}_1x_1(x_2 - x_3)x_5 + \mathrm{c}_1x_1(x_2 - x_3)ba^{-1}$, or

$$
\sigma = \psi + \phi ba^{-1}. \qquad (54)
$$

Now σ is that function of the roots of the quintic (27) which ψ is of the general quintic. As a special case of (53), therefore, we have this modified form of the sinister resolvent:

$$
\left.
\begin{aligned}
& \sigma^6 + 100(4\gamma\varepsilon - 3\delta^2)\sigma^4 + 2000(8\gamma^2\delta\zeta + 20\gamma^2\varepsilon^2 - 2\gamma\zeta^2 - 40\gamma\delta^2\varepsilon + 6\delta\varepsilon\zeta \\
& + 15\delta^4 - 4\varepsilon^3)\sigma^2 \mp 800\zeta\sqrt{(5\mathrm{D})}\,\sigma + 40000(\zeta l_1 - 25c_3^2) = 0.
\end{aligned}
\right\} \qquad (55)
$$

---

* The wrong sign is given to this term in Faà de Bruno's table.

† I have devised formulæ for deducing the values of t and v from ψ, but they are omitted as circuitous and, in comparison, useless.

Here D, l_1, and C_3 have the same values as in (19) and (4). The result thus obtained corresponds closely to (47). Let us write (55) in this form:

$$\sigma^6 + 100e_2\sigma^4 + 2000e_4\sigma^2 \mp 800\zeta\sqrt{(5\mathrm{D})}\,\sigma + 40000e_6 = 0. \tag{56}$$

This corresponds to (48). Let

$$s = \frac{1}{500}\sigma^2. \tag{57}$$

This definition corresponds to (38). Then $\sigma = 2\sqrt{5}\,(5s^{\frac{1}{2}})$, and (56) becomes, after dividing by 40000,

$$\tfrac{1}{5}(5s^{\frac{1}{2}})^6 + e_2(5s^{\frac{1}{2}})^4 + e_4(5s^{\frac{1}{2}})^2 \mp \tfrac{1}{5}\zeta\sqrt{\mathrm{D}}\,(5s^{\frac{1}{2}}) + e_6 = 0. \tag{58}$$

Transposing the odd power, and squaring, we have, as a final variation of the sinister resolvent, this sextic in s:

$$\left.\begin{aligned} 5^{10}s^6 + 2.5^9e_2s^5 + (2.5^7e_4 + 5^8e_2^2)s^4 + 2\,(5^5e_6 + 5^6e_2e_4)\,s^3 \\ + (2.5^4e_2e_6 + 5^4e_4^2)\,s^2 + (2.5^2e_4e_6 - \zeta^2\mathrm{D})\,s + e_6^2 = 0. \end{aligned}\right\} \tag{59}$$

The last two equations correspond respectively to (49) and (50).

25. The third function of the roots named in paragraph 21 as suitable for a resolvent is of all three the easiest to deal with. Let

$$\tau = \psi\phi^{-1}, \tag{60}$$

$$t = \sigma\phi^{-1} = \tau + ba^{-1}. \tag{61}$$

Thus $\tau = \dfrac{\mathrm{C}_1x_1(x_2 - x_3)\,x_5}{\mathrm{C}_1x_1(x_2 - x_3)}$. If we divide both numerator and denominator by $x_1x_2x_3x_4x_5$, we have $\tau = \dfrac{\mathrm{C}_1x_3^{-1}x_4^{-1} - \mathrm{C}_1x_2^{-1}x_4^{-1}}{\mathrm{C}_1x_3^{-1}x_4^{-1}x_5^{-1} - \mathrm{C}_1x_2^{-1}x_4^{-1}x_5^{-1}}$. Now $\mathrm{C}_1x_3^{-1}x_4^{-1} = \mathrm{C}_1x_1^{-1}x_2^{-1}$, $\mathrm{C}_1x_2^{-1}x_4^{-1} = \mathrm{C}_1x_1^{-1}x_3^{-1}$, $\mathrm{C}_1x_3^{-1}x_4^{-1}x_5^{-1} = \mathrm{C}_1x_1^{-1}x_2^{-1}x_5^{-1}$, $\mathrm{C}_1x_2^{-1}x_4^{-1}x_5^{-1} = \mathrm{C}_1x_1^{-1}x_3^{-1}x_5^{-1}$. Hence $\tau = \dfrac{\mathrm{C}_1x_1^{-1}(x_2^{-1} - x_3^{-1})}{\mathrm{C}_1x_1^{-1}(x_2^{-1} - x_3^{-1})x_5^{-1}}$, the reciprocal of the same function of the reciprocals of the roots. It is therefore a covariant function, and having six values is to be determined by a covariant sextic, say

$$(\phi_1\tau - \psi_1)(\phi_2\tau - \psi_2)(\phi_3\tau - \psi_3)(\phi_4\tau - \psi_4)(\phi_5\tau - \psi_5)(\phi_6\tau - \psi_6) = 0. \tag{62}$$

To determine the coefficients we need to know but one, either the first or the last, the rest being obtainable at once by the usual rules for the formation of covariants. Considering (62) under this form,

$$\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\tau^6 + \ldots + \psi_1\psi_2\psi_3\psi_4\psi_5\psi_6 = 0, \tag{63}$$

we see that this is the correct form of the covariant, the leading coefficient being a function of the roots of twelve dimensions, the final eighteen, and the weight eighteen. The covariant is therefore of the sixth degree and sixth order. Examining the list of covariants of the quintic, we find that any covariant of the sixth degree and sixth order must be of the form $m_1\mathrm{AL} + m_2\mathrm{Q}^3 + m_3\mathrm{BQ} + m_4\mathrm{C}^2$, where m_1, m_2, m_3, m_4, are unknown numerical multipliers, A is the quintic, L its simplest linear, Q its simplest quadratic, C its simplest cubic, and B its simplest biquadratic or quartic covariant. Ascertaining in any manner the value of $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, we find that $m_1 = 1$, $m_2 = m_3 = 0$, $m_4 = -25$. This can be done by Mr. Cayley's method, by symmetric functions, assuming that one root vanishes, or by discussion of the Eulerian equations, as will be shown further on. The latter method is no doubt the simpler. We thus derive the *central resolvent*,

$$\mathrm{AL} - 25\mathrm{C}^2 = 0. \tag{64}$$

Here

$$\mathrm{A} = a\tau^5 + 5b\tau^4 + 10c\tau^3 + 10d\tau^2 + 5e\tau + f,$$
$$\mathrm{L} = l_0'\tau + l_1',$$
$$\mathrm{C} = c_0'\tau^3 + c_1'\tau^2 + c_2'\tau + c_3',$$

where the coefficients have the same values as in paragraph 23. If we obtain a value of τ by this resolvent, the corresponding value of t is at once found by (61). In most cases it will be easier to use a variation of (64), being that special case of it which results when the quintic has the form given in (27). In this case $\tau = t$, and we have

$$\mathrm{A}_y\mathrm{L}_y - 25\mathrm{C}_y^2 = 0. \tag{65}$$

This is the resolvent (3) explained and exemplified near the beginning.

26. Of the different forms which the dexter and sinister resolvents may assume, (47) and (55) are usually most convenient in practice. To illustrate them we may take one of the examples (15) employed for the central resolvent (3). In that case we have, for determining ϕ by the dexter resolvent,

$$\phi^6 + 37\phi^4 + 115\phi^2 \pm 1504\phi - 249 = 0, \tag{66}$$

and for determining σ by the sinister resolvent,

$$\sigma^6 + 4\sigma^4 - 32\sigma^2 \pm 1504\sigma + 1408 = 0. \tag{67}$$

A value of ϕ by (66) is ∓ 3, and a value of σ by (67) is ∓ 4. It happens that these values are consistent with each other, as is shown by the result of (16), $t = \frac{4}{3}$, but we could not assume it without some such verification. There is little choice in point of convenience between the dexter and sinister resolvents. As between either and the central resolvent, it may be remarked that with the latter the chief difficulty lies in the calculation of l_0 and l_1, while either of the others requires the calculation of one of these quantities, besides the more laborious evaluation of the discriminant, though on the other hand there are fewer coefficients to be determined. Were nothing further involved, we might find one as convenient as another; but it is necessary to determine two quantities, t and v or t and s, and we shall see that on the whole, both for theoretic symmetry and practical availability, the preference is to be given to the central resolvent.

27. Observing (38), (57), and (60), we see that the three quantities t, v, and s, obtainable by the central, dexter, and sinister resolvents respectively, are connected by the necessary relation

$$s^{\frac{1}{2}} = tv^{\frac{1}{2}}. \tag{68}$$

For each value of t, six in all, there are corresponding values of v and s, and *vice versa.* By the simultaneous use of all three resolvents, with (60), the values corresponding to each other could be picked out; but the determination and minute comparison of eighteen roots would be most laborious. Means of determining, from a single root of one resolvent, any other necessary quantities, are therefore requisite, but before entering on that question it is desirable to inquire which of these quantities are needed for expressing the elements.

28. From (41) we know that for expressing the fifth powers of the elements it is sufficient to know the values of the combinations u_1u_4, u_2u_3, $u_1^2u_3$, $u_2^2u_1$, $u_3^2u_4$, $u_4^2u_2$. We shall shortly find that the relations stated in (40) and (42) enable us to express these combinations in terms of the twelve-valued quantities $v^{\frac{1}{2}}$ and $s^{\frac{1}{2}}$ and of the coefficients. We need a value of $v^{\frac{1}{2}}$, and a corresponding value of $s^{\frac{1}{2}}$. It is obvious from (68) that we get these when we have corresponding values of v and t, or of s and t, all six-valued functions. We may therefore dismiss either v or s from further consideration. Let us retain v as the simpler function of the two, merely remembering that in all places where v occurs it may be replaced by st^{-2}.

29. That the $s^{\frac{1}{2}}$ appearing in (42) is the same quantity as the $s^{\frac{1}{2}}$, equal to $tv^{\frac{1}{2}}$, defined in (57), is proved on replacing the elements in (42) by their equivalents as given in (28). By (40) and (42),

$$\left.\begin{array}{ll} u_1u_4 = -\gamma + v^{\frac{1}{2}}, & u_2u_3 = -\gamma - v^{\frac{1}{2}} \\ u_1^2u_3 + u_4^2u_2 = -\delta - tv^{\frac{1}{2}}, & u_2^2u_1 + u_3^2u_4 = -\delta + tv^{\frac{1}{2}}. \end{array}\right\} \tag{69}$$

Hence

$$\left.\begin{array}{l} 2u_1^2u_3 = -\delta - tv^{\frac{1}{2}} + (u_1^2u_3 - u_4^2u_2), \\ 2u_4^2u_2 = -\delta - tv^{\frac{1}{2}} - (u_1^2u_3 - u_4^2u_2), \\ 2u_2^2u_1 = -\delta + tv^{\frac{1}{2}} + (u_2^2u_1 - u_3^2u_4), \\ 2u_3^2u_4 = -\delta + tv^{\frac{1}{2}} - (u_2^2u_1 - u_3^2u_4). \end{array}\right\} \tag{70}$$

Multiplying the first pair together, and also the second pair, and remembering that $u_1^2u_4^2u_2u_3 = (\gamma^2 - v)(-\gamma + v^{\frac{1}{2}})$, $u_1u_4u_2^2u_3^2 = (\gamma^2 - v)(-\gamma - v^{\frac{1}{2}})$, we find that

$$\left.\begin{aligned} (u_1^2u_3 - u_4^2u_2)^2 &= (\delta + tv^{\frac{1}{2}})^2 + 4(\gamma^2 - v)(\gamma - v^{\frac{1}{2}}), \\ (u_2^2u_1 - u_3^2u_4)^2 &= (\delta - tv^{\frac{1}{2}})^2 + 4(\gamma^2 - v)(\gamma + v^{\frac{1}{2}}). \end{aligned}\right\} \qquad (71)$$

If the square roots of these quantities be substituted in (70), the combinations forming the left hand members are expressed in terms of t, $v^{\frac{1}{2}}$, and the coefficients. Then, since

$$\left.\begin{aligned} u_1^5 &= (u_1^2u_3)^2\, u_2^2u_1\, (u_2u_3)^{-2}, & u_2^5 &= (u_2^2u_1)^2\, u_4^2u_2\, (u_1u_4)^{-2}, \\ u_4^5 &= (u_4^2u_2)^2\, u_3^2u_4\, (u_2u_3)^{-2}, & u_3^5 &= (u_3^2u_4)^2\, u_1^2u_3\, (u_1u_4)^{-2}, \end{aligned}\right\} \qquad (72)$$

we have all that is needed to express the fifth-powers of the elements in terms of t, $v^{\frac{1}{2}}$, and the coefficients. There is nothing new, except the quantity t and the use of a symbol for v, in this paragraph, which merely states the reasoning which justifies the formulæ of Signor Brioschi. We can, however, improve upon these expressions, which are simple only in appearance.

30. We have seen in (35) the form of the fifth-powers of the elements as arrived at by Lagrange and Vandermonde, after investigating those powers as functions of the roots. If we put $2p_1 = u_1^5 + u_4^5$, $2p_2 = u_2^5 + u_3^5$, $2q_1 = u_1^5 - u_4^5$, $2q_2 = u_2^5 - u_3^5$, we have

$$\left.\begin{aligned} u_1^5 &= p_1 + q_1, & u_2^5 &= p_2 + q_2, \\ u_4^5 &= p_1 - q_1, & u_3^5 &= p_2 - q_2. \end{aligned}\right\} \qquad (73)$$

Multiplying by pairs, we have $q_1^2 = p_1^2 - u_1^5u_4^5$, $q_2^2 = p_2^2 - u_2^5u_3^5$. This is substantially the form reached by Sir James Cockle.[*] If we combine (35) and (73) by putting $2p_1 + 2p_2 = \Sigma u^5 = r_1$, $2p_1 - 2p_2 = r_2$, we find that $p_1 = \frac{1}{4}r_1 + \frac{1}{4}r_1$, and $p_2 = \frac{1}{4}r_1 - \frac{1}{4}r_2$. It remains to determine the values ef r_1 and r_2 in terms of t, $v^{\frac{1}{2}}$, and the coefficients. We are obviously able to do this by substitution, observing (72). The results, however, are unnecessarily complex, and to simplify them we shall find it advantageous to resort to the Eulerian equations (31) and (32).

31. For brevity in handling the expressions in (70), let $2m_1 = -\delta - tv^{\frac{1}{2}}$, $2m_2 = -\delta + tv^{\frac{1}{2}}$, $2n_1 = u_1^2u_3 - u_4^2u_2$, $2n_2 = u_2^2u_1 - u_3^2u_4$. Then $u_1^2u_3 = m_1 + n_1$, $u_4^2u_2 = m_1 - n_1$, $u_2^2u_1 = m_2 + n_2$, $u_3^2u_4 = m_2 - n_2$. Since $u_1u_2u_3u_4 = u_1u_4 . u_2u_3 = (-\gamma + v^{\frac{1}{2}})(-\gamma - v^{\frac{1}{2}}) = \gamma^2 - v$, we may write the third Eulerian equation (31) thus:

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}\left[u_1^4u_2^2u_3u_4 + u_4^4u_3^2u_1u_2 + u_2^4u_4^2u_1u_3 + u_3^4u_1^2u_2u_4\right]. \qquad (74)$$

The expressions within the bracket are severally equal to $(m_1 + n_1)(m_2 + n_2)$

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[*] "On the Resolution of Quintics," Quarterly Journal, June, 1860.

$(-\gamma + v^{\frac{1}{2}})$, $(m_1 - n_1)(m_2 - n_2)(-\gamma + v^{\frac{1}{2}})$, $(m_1 - n_1)(m_2 + n_2)(-\gamma - v^{\frac{1}{2}})$, $(m_1 + n_1)(m_2 - n_2)(-\gamma - v^{\frac{1}{2}})$, and their sum is $-4\gamma m_1 m_2 + 4n_1 n_2 v^{\frac{1}{2}}$. Therefore

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}(\gamma t^2 v - \gamma\delta^2 + p), \tag{75}$$

where $p = v^{\frac{1}{2}}(u_1^2 u_3 - u_4^2 u_2)(u_2^2 u_1 - u_3^2 u_4)$, and hence also

$$p = \gamma\delta^2 - \gamma t^2 v + (v - \gamma^2)(\varepsilon - \gamma^2 - 3v). \tag{76}$$

Again, we may write the fourth Eulerian equation (32) thus:

$$\begin{aligned} \Sigma u^5 &= -\zeta - 20\gamma\delta + 10\,[(m_1+n_1)\,u_1u_4 + (m_1 - n_1)\,u_1u_4 + (m_2 + n_2)\,u_2u_3 + (m_2 - n_2)u_2u_3] \\ &= -\zeta - 20\gamma\delta + 10[(-\delta - tv^{\frac{1}{2}})(-\gamma + v^{\frac{1}{2}}) + (-\delta + tv^{\frac{1}{2}})(-\gamma - v^{\frac{1}{2}})] \\ &= -\zeta - 20tv. \end{aligned} \tag{77}$$

This equation coincides with one given by Professor Schläfli, except, of course, that instead of tv he writes what corresponds in his notation to $v^{\frac{1}{2}}s^{\frac{1}{2}}$.

32. We are now able to proceed with the subject of paragraph 30. Since $r_1 = u_1^5 + u_2^5 + u_3^5 + u_4^5$, we have from (77)

$$r_1 = -\zeta - 20tv. \tag{78}$$

For determining r_2, equal to $u_1^5 + u_4^5 - u_2^5 - u_3^5$, we must expand these powers separately by employing (72). To avoid fractions, let us multiply each by $u_1^2 u_2^2 u_3^2 u_4^2 = (\gamma^2 - v)^2$. Thus,

$$\left.\begin{aligned} (\gamma^2 - v)^2 u_1^5 &= (m_1^2 + 2m_1n_1 + n_1^2)(m_2 + n_2)(\gamma^2 - 2\gamma v^{\frac{1}{2}} + v), \\ (\gamma^2 - v)^2 u_4^5 &= (m_1^2 - 2m_1n_1 + n_1^2)(m_2 - n_2)(\gamma^2 - 2\gamma v^{\frac{1}{2}} + v), \\ (\gamma^2 - v)^2 u_2^5 &= (m_2^2 + 2m_2n_2 + n_2^2)(m_1 - n_1)(\gamma^2 + 2\gamma v^{\frac{1}{2}} + v), \\ (\gamma^2 - v)^2 u_3^5 &= (m_2^2 - 2m_2n_2 + n_2^2)(m_1 + n_1)(\gamma^2 + 2\gamma v^{\frac{1}{2}} + v). \end{aligned}\right\} \tag{79}$$

The sum of the first two, less the sum of the last two, is (remembering that $4n_1n_2v^{\frac{1}{2}} = p$)

$$\left.\begin{aligned} (\gamma^2 - v)^2 r_2 = (m_1 + m_2)&[pv^{-\frac{1}{2}}(\gamma^2 + v) - 4m_1m_2\gamma v^{\frac{1}{2}}] \\ &+ 2(m_1 - m_2)[m_1m_2(\gamma^2 + v) - p\gamma] \\ &+ 2(\gamma^2 + v)(n_1^2 m_2 - n_2^2 m_1) - 4\gamma v^{\frac{1}{2}}(n_1^2 m_2 + n_2^2 m_1). \end{aligned}\right\} \tag{80}$$

The values of $4n_1^2$ and $4n_2^2$ are given by (71), and that of p by (76), while $m_1 + m_2 = -\delta$, $m_1 - m_2 = -tv^{\frac{1}{2}}$, $m_1m_2 = \frac{1}{4}(\delta^2 - t^2v)$. Substituting these values, and dividing throughout by $(\gamma^2 - v)^2$, we have

$$\left.\begin{aligned} r_2 = (\gamma^2 - v)^{-1}(&12\gamma tv^2 - \delta v^2 - t^3v^2 + 4\gamma^3 tv + 2\gamma^2\delta v + \gamma\delta t^2 v \\ &+ \delta^2 tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac{1}{2}}. \end{aligned}\right\} \tag{81}$$

For determining q_1 and q_2, we have—see (73)—

$$\left.\begin{aligned} q_1^2 &= (\tfrac{1}{4}r_1 + \tfrac{1}{4}r_2)^2 + (\gamma - v^{\frac{1}{2}})^5, \\ q_2^2 &= (\tfrac{1}{4}r_1 - \tfrac{1}{4}r_2)^2 + (\gamma + v^{\frac{1}{2}})^5. \end{aligned}\right\} \tag{82}$$

Let $2s_1 = q_1^2 + q_2^2$, and $2s_2 = q_1^2 - q_2^2$; then $q_1 = \surd(s_1 + s_2)$, $q_2 = \surd(s_1 - s_2)$, where

$$\left.\begin{aligned} s_1 &= \tfrac{1}{16} r_1^2 + \tfrac{1}{16} r_2^2 + \gamma^5 + 10\gamma^3 v + 5\gamma v^2, \\ s_2 &= \tfrac{1}{8} r_1 r_2 - (5\gamma^4 + 10\gamma^2 v + v^2) v^{\frac{1}{2}}. \end{aligned}\right\} \qquad (83)$$

And (73) becomes

$$\left.\begin{aligned} u_1^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 + \surd(s_1 + s_2), & u_2^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 + \surd(s_1 - s_2), \\ u_4^5 &= \tfrac{1}{4} r_1 + \tfrac{1}{4} r_2 - \surd(s_1 + s_2), & u_3^5 &= \tfrac{1}{4} r_1 - \tfrac{1}{4} r_2 - \surd(s_1 - s_2). \end{aligned}\right\} \qquad (84)$$

We have thus demonstrated the element-formulæ first given, numbered (7) to (12).

33. For the critical cases $v = \gamma^2$ and $v = 0$ separate element-formulæ were presented in paragraphs 7 and 8, and these will now be proved. For the case $v = \gamma^2$, we had $u_4 = 0$, $v^{\frac{1}{2}} = \gamma$, $u_2 u_3 = -2\gamma$. From (69) we find that $u_1^2 u_3 = -(\gamma t + \delta)$, $u_2^2 u_1 = \gamma t - \delta$. Hence, by (72), $u_1^5 = (\gamma t + \delta)^2 (\gamma t - \delta)(4\gamma^2)^{-1}$, which proves (21). Also, $u_3 = u_1^2 u_3 . u_1^{-2} = -(\gamma t + \delta) u_1^{-2}$, which proves (22). It is to be remarked that the substitution of these simpler expressions does not impugn the generality of the element-formulæ (7), which are applicable to this as to all other cases. The expression given in (9) for the quantity r_2 is indeterminate in the present instance. Its value is $u_1^5 - u_2^5 - u_3^5$. We know what u_1^5 is, and that $u_3^5 = -32\gamma^5 u_2^{-5}$. To determine u_2^5, we have $u_2^5 = u_2 u_3 (u_2^2 u_1)^2 (u_1^2 u_3)^{-1} = 2\gamma (\gamma t - \delta)^2 (\gamma t + \delta)^{-1}$. For DeMoivre's quintic, $x^5 + 10\gamma x^3 + 20\gamma^2 x + \zeta = 0$, we find $v = \gamma^2$, $t = 0$, $u_1 = u_4 = 0$. In this case $r_2 = -u_2^5 - u_3^5 = -r_1 = \zeta$, and the solution follows by (7). For Euler's quintic, $x^5 + 10\delta x^2 + 5\varepsilon x + \zeta = 0$, where $16\delta^4 + \varepsilon^3 = 2\delta\varepsilon\zeta$, we find $v = \gamma^2 = 0$, whence by (7) $u_1^5 = \tfrac{1}{2} r_1 + \tfrac{1}{2} r_2$, $u_2^5 = \tfrac{1}{2} r_1 - \tfrac{1}{2} r_2$, $u_3^5 = u_4^5 = 0$. Here $r_1 = -\zeta$. By (30) and (31), $u_2^2 u_1 = -2\delta$, $u_1^3 u_2 = -\varepsilon$, whence u_1^5 and u_2^5 are found, and $r_2 = u_1^5 - u_2^5$. The binomial quintic $x^5 + \zeta = 0$ may be regarded as a special case of DeMoivre's, with u_3 vanishing, or of Euler's, with u_2 vanishing.

34. When $v = 0$, the element-formulæ are valid, but r_2 becomes indeterminate. In this case the quantity t must be dispensed with, being usually infinite, while s, equal to $t^2 v$, is finite. Where t appears, we must substitute $s^{\frac{1}{2}} v^{-\frac{1}{2}}$. Then $tv = s^{\frac{1}{2}} v^{\frac{1}{2}} = 0$, and by (8) or (78) $r_1 = -\zeta$. Putting $v = 0$ in (79), we obtain, by appropriate summation,

$$\left.\begin{aligned} \gamma^2 r_1 &= 2(m_1 + m_2) m_1 m_2 + 4(m_1 - m_2) n_1 n_2 + 2n_1^2 m_2 + 2n_2^2 m_1, \\ \gamma^2 r_2 &= 2(m_1 - m_2) m_1 m_2 + 4(m_1 + m_2) n_1 n_2 + 2n_1^2 m_2 - 2n_2^2 m_1. \end{aligned}\right\} \qquad (85)$$

Here $r_1 = -\zeta$, $m_1 + m_2 = -\delta$, $m_1 - m_2 = -s^{\frac{1}{2}}$, $4m_1 m_2 = \delta^2 - s$, $n_1^2 = m_1^2 + \gamma^3$, $n_2^2 = m_2^2 + \gamma^3$, whence $2n_1^2 m_2 + 2n_2^2 m_1 = 2(m_1 + m_2)(m_1 m_2 + \gamma^3)$, and $2n_1^2 m_2 - 2n_2^2 m_1 = 2(m_1 - m_2)(m_1 m_2 - \gamma^3)$. Substituting these values,

$$\left.\begin{aligned} -\gamma^2\zeta &= -\delta(\delta^2 - s + 2\gamma^3) - 4n_1n_2s^{\frac{1}{2}} \\ \gamma^2 r_2 &= -s^{\frac{1}{2}}(\delta^2 - s - 2\gamma^3) - 4n_1n_2\delta. \end{aligned}\right\} \tag{86}$$

From (76), since $p = 4n_1n_2v^{\frac{1}{2}} = 0$, we obtain

$$s = \gamma^3 - \gamma\varepsilon + \delta^2. \tag{87}$$

Eliminating n_1n_2 from (86),

$$r_2 = s^{-\frac{1}{2}}(3\gamma^4 - 4\gamma^2\varepsilon + 4\gamma\delta^2 - \delta\zeta + \varepsilon^2). \tag{88}$$

We have thus demonstrated the special element-formulæ of paragraph 8, except for the case $s = 0$. We may express (88) more briefly thus:

$$r_2 = s^{-\frac{1}{2}}(4\gamma s - \gamma^4 - \delta\zeta + \varepsilon^2). \tag{89}$$

When $s = 0$, we have from (86) $r_2 = -4n_1n_2\gamma^{-2}\delta$; also, by multiplying together the two equations (71), $16n_1^2n_2^2 = (\delta^2 + 4\gamma^3)^2$, or $4n_1n_2 = \delta^2 + 4\gamma^3$; so that $r_2 = -\gamma^2\delta^3 - 4\gamma\delta$, as stated in paragraph 8. The contrary sign might be given to $4n_1n_2$ without affecting the elements otherwise than by changing their order.

35. Results not uninteresting may be obtained by assimilating the element-formulæ to the forms of the fifth roots of unity. We know that r_1 is a rational function of the roots of the quintic, and since $500v = \phi^2$ we see that $r_2\sqrt{5}$ is another. The quantities q in (35) were found by Lagrange to be of the form $a\sqrt{(-10 \pm 2\sqrt{5})} \pm b\sqrt{(-10 \mp \sqrt{5})}$, where a and b are rational functions of the roots; and this may be verified by expanding u^5 in terms of the roots. Taking (14) as an example, we may replace $\sqrt{(-\frac{146}{5} \pm \frac{1018}{125}\sqrt{5})}$ by $\frac{36}{25}\sqrt{(-10 \pm 2\sqrt{5})} \mp \frac{23}{25}\sqrt{(-10 \mp 2\sqrt{5})}$. The values of a and b may be readily ascertained from another form, which will be found equivalent to that just given as having been indicated by Lagrange, namely, $(a - \frac{1}{2}b \mp \frac{1}{2}b\sqrt{5})\sqrt{(-10 \pm 2\sqrt{5})}$. Let us denote $a - \frac{1}{2}b$ by h_1, and $-\frac{1}{2}b\sqrt{5}$ by h_2; then h_1 and $h_2\sqrt{5}$ are rational functions of the roots, and it only remains to determine their values. We have $(h_1 \pm h_2)\sqrt{(-10 \pm 2\sqrt{5})} = \sqrt{(s_1 \pm s_2)}$, whence $(h_1^2 \pm 2h_1h_2 + h_2^2)(-10 \pm 2\sqrt{5}) = s_1 \pm s_2$, and $(h_1^2 - h_2^2)4\sqrt{5} = \sqrt{(s_1^2 - s_2^2)}$. From these equations we find that

$$\left.\begin{aligned} 400h_1^2 &= 10\sqrt{(5s_1^2 - 5s_2^2)} - 25s_1 - 5\sqrt{5}s_2, \\ 400h_2^2 &= -10\sqrt{(5s_1^2 - 5s_2^2)} - 25s_1 - 5\sqrt{5}s_2. \end{aligned}\right\} \tag{90}$$

Taking the same example (14) as before, we obtain $h_1^2 = (\frac{19}{10})^2$, $h_2^2 = (\frac{23}{50}\sqrt{5})^2$. The quantities q may therefore be represented by $(\frac{19}{10} \pm \frac{23}{50}\sqrt{5})\sqrt{(-10 \pm 2\sqrt{5})}$. A similar modification, subject to the same remarks, may be made in the form of n_1 and n_2, which have the form $\sqrt{(s_3 \pm s_4)}$, as may be seen from (71), where

s_3 and $s_4\sqrt{5}$ are rational functions of the roots. The values of these symbols are

$$\left.\begin{aligned} s_3 &= \tfrac{1}{4}\delta^2 + \tfrac{1}{4}t^2v + \gamma^3 - \gamma v, \\ s_4 &= (\tfrac{1}{2}\delta t - \gamma^2 + v)\,v^{\frac{1}{2}}. \end{aligned}\right\} \qquad (91)$$

The quantities n may be put under the form $(h_3 \pm h_4)\sqrt{(-10 \pm 2\sqrt{5})}$, by formulæ precisely similar to (90).

36. Assuming the coefficients of the quintic to be rational, t and v may be rational or irrational, but each is normally a rational function of the other, by Lagrange's theory, since each has six values respectively corresponding, and we shall, in fact, see each expressed rationally in terms of the other. Any rational function of t or v is a rational function of the roots, but the contrary does not hold. The quantities a and b, h_1 and $h_2\sqrt{5}$, for example, discussed in the preceding paragraph, are not necessarily rational functions of t and v. But by (76) p, or $4n_1n_2v^{\frac{1}{2}}$, is such a function, and so, it may be shown, is $q_1q_2v^{\frac{1}{2}}$, which let us denote by k. We note first that $n_2 = 4p^{-1}n_2^2v^{\frac{1}{2}}n_1$, and therefore is of the form $n_2 = (a_1 + a_2v^{\frac{1}{2}})\,n_1$, where a_1 and a_2 are rational functions of t and v. If we expand $q_1 = \frac{1}{2}(u_1^5 - u_4^5)$ by means of (79), it will take the form $q_1 = (b_1 + b_2v^{\frac{1}{2}})\,n_1$, where b_1 and b_2 are rational functions of t and v. Similarly, $q_2 = (b_1 - b_2v^{\frac{1}{2}})\,n_2$; hence $4k = p\,(b_1^2 - b_2^2v)$, and k is a rational function as stated.

37. Let $\mathrm{E} = \sqrt{(s_1^2s_2^{-2} - 1)} = ks_2^{-1}v^{-\frac{1}{2}}$, $\mathrm{Z} = \mathrm{E}^2 + 1 = s_1^2s_2^{-2}$, $\mathrm{H} = s_1^{-1}s_2^2$, $\mathrm{B} = \frac{1}{4}r_1$, $\mathrm{B}' = \frac{1}{4}r_2s_1^{-1}s_2$; then $\frac{1}{4}r_2 = \mathrm{B}'\mathrm{Z}^{\frac{1}{2}}$, $s_1 = \mathrm{HZ}$, $s_2 = \mathrm{HZ}^{\frac{1}{2}}$, and if these be substituted in (7) we have

$$u_1^5 = \mathrm{B} + \mathrm{B}'\mathrm{Z}^{\frac{1}{2}} + \sqrt{(\mathrm{HZ} + \mathrm{HZ}^{\frac{1}{2}})}. \qquad (92)$$

The quantities here introduced seem arbitrary, and are in reality complex in character, yet they will all be recognized as rational functions of t and v. That the fifth-powers of the elements can be exhibited in the form shown in (92) was ascertained by Mr. G. P. Young, by a minute consideration of the possible surds involved in the roots of the general quintic.* The quantities B, B', H, E, were shown by him to be rational functions, of unknown form, of each other. He indicated no means of determining them by a resolvent, nor did the quantities t and v enter into his discussion. We have now found the values of his undetermined symbols. Taking (12) as an example, where $v = \frac{1}{20}$, $r_1 = 90$, $r_2 = 6\sqrt{5}$, $s_1 = -\frac{146}{5}$, $s_2 = -\frac{1018}{125}\sqrt{5}$, it will be found that $\mathrm{B} = \frac{45}{2}$, $\mathrm{B}' = \frac{3054}{365}$, $\mathrm{Z} = \frac{666125}{259081}$, and $\mathrm{H} = -\frac{518162}{45625}$. By the same investigation, Mr. Young arrived at the following form, without determining the values of the quantities:

$$u_1^2u_3 = -\tfrac{1}{2}\delta + \mathrm{CZ}^{\frac{1}{2}} + (\Theta + \Phi\mathrm{Z}^{\frac{1}{2}})\sqrt{(\mathrm{HZ} + \mathrm{HZ}^{\frac{1}{2}})}. \qquad (93)$$

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* See note to paragraph 8.

We observe that $\mathrm{c}z^{\frac{1}{5}} = -\frac{1}{2}tv^{\frac{1}{5}}$, so that $\mathrm{c} = -\frac{1}{2}ts_1^{-1}s_2v^{\frac{1}{5}}$. Using the same illustration, where $t = 30$, it follows that $\mathrm{c} = -\frac{1527}{730}$.

38. Let us now consider the most important question remaining, namely, how to determine a value of v from the corresponding value of t, and the contrary. The theory of Lagrange, elaborated by Meyer Hirsch, supplies formulæ for the value of an n-valued function of roots in terms of the corresponding value of another n-valued function. It would be more troublesome, however, to pursue that method than to make use of the Eulerian equations in the manner which will now be exhibited.

39. The product of the left hand members of the equations numbered (71), multiplied by v, is the same as p^2, as defined in paragraph 31. Hence,

$$p^2 = (\delta^2 - t^2v)^2v + 8(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2v + 2\delta tv)v + 16(\gamma^2 - v)^3v. \qquad (94)$$

If we set this equal to the square of p as given in (76), we see at once that the resulting equation is divisible by $(\gamma^2 - v)$. This done,

$$\left.\begin{array}{l} 25v^3 + (-t^4 + 14\gamma t^2 + 16\delta t - 35\gamma^2 - 6\varepsilon)v^2 \\ \quad + (-2c_0t^2 + 2\gamma\delta^2 + 4\gamma^2\varepsilon + 11\gamma^4 + \varepsilon^2)v - c_0^2 = 0. \end{array}\right\} \qquad (95)$$

Here $c_0 = -\gamma^3 + \gamma\varepsilon - \delta^2$, as in paragraph 2. Again, summing the expressions in (79), and dividing by $(\gamma^2 - v)$, we find that

$$\Sigma u^5 = (\gamma^2 - v)^{-1}[\gamma t^3v - \delta t^2v + \varepsilon tv + \gamma c_0t - 10\gamma^2tv - 5tv^2 + \delta^3 - 2\gamma\delta\varepsilon] = 0. \quad (96)$$

Equating this value of Σu^5 with that in (77), and clearing of fractions, we find that

$$25tv^2 + (-\gamma t^3 + \delta t^2 - \varepsilon t - 10\gamma^2t + \zeta)v - \gamma c_0t + c_3 = 0. \qquad (97)$$

Here $c_3 = -\gamma^2\zeta + 2\gamma\delta\varepsilon - \delta^3$, as in paragraph 2. The two equations in t and v, (95) and (97),* are substantially equivalent to two of Signor Brioschi's and to two of Professor Schläfli's, though more simply deduced and expressed, owing to the advantages secured by employing the quantities t and v. By substituting st^{-2} for v, equations in t and s may be obtained upon occasion. The equations in t and s will be found useful for determining the value of t in any case where s has been ascertained by the use of the sinister resolvent. I have throughout considered it unnecessary to state separately the formulæ in terms of t and s which may be derived from those in t and v by replacing v by st^{-2}.

40. By subtracting (95) multiplied by t from (97) multiplied by v there results a second quadratic in v, say $b_0v^2 + b_1v + b_2 = 0$, where $b_0 = t^5 - 15\gamma t^3 - 15\delta t^2 + 25\gamma^2t + 5\varepsilon t + \zeta$, $b_1 = 2c_0t^3 - \gamma\delta^2t - 5\gamma^2\varepsilon t - 10\gamma^4t - \varepsilon^2t + c_3$, $b_2 = c_0^2t$.

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* Vol. VI, pp. 314, 315. I supposed them entirely new.

We may write (97) as $a_0v^2 + a_1v + a_2 = 0$. From these two quadratics we have by elimination

$$v = \frac{a_2b_0 - a_0b_2}{a_0b_1 - a_1b_0} = \frac{a_1b_2 - a_2b_1}{a_2b_0 - a_0b_2}, \tag{98}$$

$$(a_2b_0 - a_2b_0)^2 + (a_1b_0 - a_0b_1)(a_1b_2 - a_2b_1) = 0. \tag{99}$$

The resultant (99) contains, as extraneous factors, the quantity t just introduced into one equation, and also the expression in t corresponding to the factor $\gamma^2 - v$, previously introduced. From (96) we find that when $\gamma^2 - v = 0$, $\gamma^3t^3 - \gamma^2\delta t^2 + (2\gamma^2\varepsilon - 16\gamma^4 - \gamma\delta^2)t - 2\gamma\delta\varepsilon + \delta^3 = 0$. Dividing (99), after expansion, by this extraneous factor, and by t, we obtain the central resolvent in t (3) already repeatedly mentioned. It was in this manner that I first obtained that resolvent,* but it may be demonstrated with much less labor, as follows.

41. Let us resume the discussion of the central resolvent as we left it when stating (64). The mere definition of τ as a function of the roots was sufficient to demonstrate the resolvent, assuming as determined the quantities m in the equation

$$m_1\text{AL} + m_2\text{Q}^3 + m_3\text{BQ} + m_4\text{C}^2 = 0. \tag{100}$$

To determine these it will suffice to take $a = 1$, $b = d = f = 0$, whence $\tau = t$, $\gamma = c$, $\varepsilon = e$, $\delta = \zeta = 0$. In this case, referring to the tables of covariants, the leading coefficient of AL is $-15\gamma^4\varepsilon + 14\gamma^2\varepsilon^2 + \varepsilon^3$; that of Q^3 is $27\gamma^6 + 27\gamma^4\varepsilon + 9\gamma^2\varepsilon^2 + \varepsilon^3$; that of BQ is $9\gamma^6 - 9\gamma^4\varepsilon - \gamma^2\varepsilon^2 + \varepsilon^3$; and that of C^2 is $\gamma^6 - 2\gamma^4\varepsilon + \gamma^2\varepsilon^2$. In (99) we have $a_0 = 25t$, $a_1 = -\gamma t^3 - 10\gamma^2 t - \varepsilon t$, $a_2 = (-\gamma^4 + \gamma^2\varepsilon)t$, $b_0 = t^5 - 15\gamma t^3 + (25\gamma^2 + 5\varepsilon)t$, $b_1 = 2(\gamma\varepsilon - \gamma^3)t^3 - (10\gamma^4 + 5\gamma^2\varepsilon + \varepsilon^2)t$, $b_2 = (\gamma^6 - 2\gamma^4\varepsilon + \gamma^2\varepsilon^2)t$. The highest power of t in (99) which does not vanish is t^{10}, and the only terms of (99) in which t^{10} can appear are $a_2^2b_0^2 + a_1^2b_2b_0 - a_1a_2b_0b_1$. In this expression the coefficient of t^{10} is $-25\gamma^9 + 35\gamma^7\varepsilon - 11\gamma^5\varepsilon^2 + \gamma^3\varepsilon^3$. This is, as we have seen, to be divided by the extraneous factor γ^3, and the result must coincide with the leading coefficient of (100). Therefore

$$\left.\begin{aligned} (27m_2 + 9m_3 + m_4)\gamma^6 &= -25\gamma^6, \\ (-15m_1 + 27m_2 - 9m_3 - 2m_4)\gamma^4\varepsilon &= 35\gamma^4\varepsilon, \\ (14m_1 + 9m_2 - m_3 + m_4)\gamma^2\varepsilon^2 &= -11\gamma^2\varepsilon^2, \\ (m_1 + m_2 + m_3)\varepsilon^3 &= \varepsilon^3. \end{aligned}\right\} \tag{101}$$

From these, $m_1 = 1$, $m_2 = m_3 = 0$, $m_4 = -25$, as assumed in paragraph 25. The central resolvent is thus proved to have been correctly stated in (64).

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* Vol. VI, p. 315.

42. Either of the two expressions in (98) will determine a value of v when the corresponding value of t is once known. Simpler formulæ, presented in (5), remain to be demonstrated. Expansion of both sides suffices to establish the identity

$$\text{L}_y(a_2b_0-a_0b_2)+25\text{C}_y(a_1b_2-a_2b_1)=a_2(\text{A}_y\text{L}_y-25\text{C}_y^2). \tag{102}$$

The second member vanishes, by (65), and by (98) we know that $a_1b_2-a_2b_1 = v(a_2b_0-a_0b_2)$. Combining these, we derive the formulæ (5) for expressing v in terms of t,

$$v=-\tfrac{1}{25}\text{L}_y\text{C}_y^{-1}=-\text{C}_y\text{A}_y^{-1}. \tag{103}$$

In the statement of (27) it will be seen that the factor a was dropped from the original quantic, so that $\text{A}_y=\text{A}a^{-1}$, and hence also $\text{C}_y=\text{C}a^{-3}$, $\text{L}_y=\text{L}a^{-5}$; wherefore the corresponding expressions of v in terms of τ are

$$v=-\tfrac{1}{25}\text{LC}^{-1}a^{-2}=-\text{CA}^{-1}a^{-2}. \tag{104}$$

43. We might similarly eliminate t from (95) and (97), dividing the resultant by the extraneous factor $(v-\gamma^2)^3v^6$, and so obtaining the dexter resolvent (50);* or, after substituting st^{-2} for v in the same equations, we might eliminate t to produce the sinister resolvent (59).

44. To obtain an expression for t in terms of v, we may consider (95) as briefly written $a_0t^4+a_2t^2+a_3t+a_4=0$, and (97) as $b_0t^3+b_1t^2+b_2t+b_3=0$. By the usual theory for the determination of common roots,

$$t=\text{Numerator}\div\text{Denominator} \tag{105}$$

where the numerator and denominator are expressed by determinants thus:

$$\begin{array}{cc}
\text{Numerator.} & \text{Denominator.}\\
\begin{vmatrix}
a_0 & 0 & a_2 & a_3 & 0\\
0 & a_0 & 0 & a_2 & a_4\\
b_0 & b_1 & b_2 & b_3 & 0\\
0 & b_0 & b_1 & b_2 & 0\\
0 & 0 & b_0 & b_1 & b_3
\end{vmatrix}
&
\begin{vmatrix}
-a_0 & 0 & a_2 & a_3 & a_4\\
0 & a_0 & 0 & a_2 & a_3\\
-b_0 & b_1 & b_2 & b_3 & 0\\
0 & b_0 & b_1 & b_2 & b_3\\
0 & 0 & b_0 & b_1 & b_2
\end{vmatrix}
\end{array}$$

Dividing both, after evaluation, by $(v-\gamma^2)^2v^5$, we obtain these formulæ,

$$\left.\begin{aligned}
\text{Numerator} &= 5^4(2\gamma\delta+\zeta)v^2-5^2(20\gamma^3\delta+22\gamma^2\zeta-36\gamma\delta\varepsilon+24\delta^3+2\varepsilon\zeta)v\\
&\quad+50\gamma^5\delta-59\gamma^4\zeta+20\gamma^3\delta\varepsilon+40\gamma^2\delta^3\\
&\quad+42\gamma^2\varepsilon\zeta-48\gamma\delta^2\zeta-38\gamma\delta\varepsilon^2+44\delta^3\varepsilon-\delta\zeta^2+\varepsilon^2\zeta,\\
\text{Denominator} &= -5^6v^3-3.5^4d_2v^2-5^2d_4v+d^6.
\end{aligned}\right\} \tag{106}$$

Here the symbols have the same meaning as in (48). If the dexter resolvent be used in any case, in preference to the central, these formulæ, with (7), will be found best fitted to complete the work. The proof of (20) is furnished by (106), when $v = \gamma^2$. The following expression is equivalent to (106), the origin of it being due to an examination of the minors of Professor Schläfli's determinant, referred to in paragraph 19:

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* I have actually performed this operation, and do not advise any one to repeat it unnecessarily.

$$t = \begin{vmatrix} \mathrm{H}, & \mathrm{F}, & -\mathrm{L} \\ \mathrm{G}, & \mathrm{C}, & -\delta \\ \zeta, & \delta, & \gamma \end{vmatrix} \div \begin{vmatrix} \mathrm{B}, & \mathrm{F}, & \mathrm{L} \\ \mathrm{F}, & \mathrm{C}, & \delta \\ \mathrm{L}, & \delta, & -\gamma \end{vmatrix}. \tag{107}$$

It will be remembered that the quantity denoted by t was not discussed by or (so far as appears) known to the author just mentioned.

45. Let us, for present convenience, call γ, δ, v, and t the Constituents of the quintic. I have before* pointed out that for each such set of constituents there are two conjugate equations, and that "when v and t have been determined for any quintic, we can at once construct the conjugate quintic. Equation [95] is a quadratic in ε, one value being the given value of ε, the other being that of ε in the conjugate quintic. From the latter value, that of ζ may be obtained by [97]. For example, if the given quintic be $x^5 + \frac{625}{4}x + 3750 = 0$, the conjugate is $x^5 + \frac{3125}{4}x + 3125 = 0$." Here it is assumed as known that $t = -5$, $v = \frac{125}{4}$, $\gamma = 0$, $\delta = 0$. The solution of (95), regarded as a quadratic, is given by (75) and (94), namely,

$$\varepsilon = \gamma^2 + 3v + (v - \gamma^2)^{-1}(\gamma t^2 v - \gamma\delta^2 + p), \tag{108}$$

where p is such that

$$p^2 = (\delta^2 - t^2 v)^2 v + 8\,(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2 v + 2\delta t v)v + 16\,(\gamma^2 - v)^3 v. \tag{109}$$

In the simple case taken for illustration, a trinomial of the form $x^5 + 5\varepsilon x + \zeta = 0$, this becomes

$$\varepsilon = 3v \pm \sqrt{(t^4 v - 16v^2)}. \tag{110}$$

Hence $\varepsilon = 3 . \frac{125}{4} \pm \sqrt{(5^4 . \frac{125}{4} - 5^6)} = (3 \pm 2)\,\frac{125}{4}$, namely, $\frac{125}{4}$ or $\frac{625}{4}$, as stated. The value of ζ by (97) is

$$\left.\begin{aligned} \zeta = (v - \gamma^2)^{-1}(&- 25tv^2 + \gamma t^3 v - \delta t^2 v + \varepsilon t v + 10\gamma^2 t v - \gamma^4 t \\ &+ \gamma^2 \varepsilon t - \gamma\delta^2 t - 2\gamma\delta\varepsilon + \delta^3). \end{aligned}\right\} \tag{111}$$

In the case of the trinomial in question, this becomes

$$\zeta = t\,(\varepsilon - 25v). \tag{112}$$

Hence $\zeta = -5\,(\frac{125}{4} - \frac{3125}{4}) = 3750$, or $\zeta = -5\,(\frac{625}{4} - \frac{3125}{4}) = 3125$, as stated.

46. In addition to the illustration just quoted, we may take equations (12)

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[*] Vol. VI, p. 315.

and (15), which give the following:

	γ	δ	t	v	ε	ζ
(12)	$-\frac{7}{2}$	3	30	$\frac{1}{20}$	$\frac{41 \pm 141}{250}$	$-\frac{891 \pm 141}{750}$
(15)	$-\frac{1}{10}$	$-\frac{1}{5}$	$\frac{4}{3}$	$\frac{9}{500}$	$\frac{6932 \pm 632}{5.61}$	$-\frac{200040 \pm 246480}{61^2}$

Since each quintic has six sets of constituents, it has six other quintics conjugate to it, and these have others, the only apparent limit to their formation being that they must all be of the same class as regards the number of real roots. This I infer from the fact that in (79) a change of sign in either n_1, n_2, or $v^{\frac{1}{2}}$, while not affecting the reality of the elements, will change the sign of p, and therefore lead to the elements of the conjugate. The construction of quintics from a given set of constituents is useful chiefly for obtaining examples wherein the constituents are simple in nature. To construct a quintic with rational coefficients and constituents, rational values must be found for the symbols employed in (109) such as will satisfy that equation, and the rest follows. For example, the values $\gamma = 0$, $\delta = 1$, $t = -2$, $v = 1$, $p = \pm 5$, satisfy (109), and it follows that $\varepsilon = 3 \pm 5$, $\zeta = 41 \mp 10$. Perhaps the simplest values possible for γ and δ are $\gamma = 0$, $\delta = 0$, producing the trinomial $x^5 + 5\varepsilon x + \zeta = 0$, for which (110) and (112) are available. In this case we have only to find values of the symbols consistent with the relation embodied in (110), $p^2 = t^4 v - 16v^2$, or

$$v = t^4 (p^2 v^{-2} + 16)^{-1}, \tag{113}$$

and for this we may assign any values we please to the quantities t and pv^{-1}.*

47. For the trinomial just spoken of, $x^5 + 5\varepsilon x + \zeta = 0$, the element-formulæ (7) become most readily applicable. In this case

$$\left.\begin{array}{ll} r_1 = -\zeta - 20tv, & r_2 = t^3 v^{\frac{1}{2}}, \\ s_1 = \frac{1}{16}(r_1^2 + r_2^2), & s_2 = \frac{1}{8} r_1 r_2 - v^{\frac{5}{2}}. \end{array}\right\} \tag{114}$$

Let us take, for example, the case quoted in paragraph 45, wherein $t = -5$, $pv^{-1} = -2$, $v = \frac{125}{4}$, $\varepsilon = 3v + p = \frac{125}{4}$, $\zeta = t(\varepsilon - 25v) = 3750$. Then $r_1 = -5^4$, $r_2 = -\frac{1}{2} 5^4 \sqrt{5}$, $s_1 = \frac{9}{64} 5^8$, $s_2 = \frac{9}{32} 5^7 \sqrt{5}$, and these values, substituted in (7), complete the element-formulæ.

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* The construction of trinomials of this form by assigning values to quantities identifiable with the two here named has been effected by Mr. G. P. Young (*American Journal of Mathematics*, VII, 170) in a different manner—the quantity v not appearing at all, and the quantity t being introduced as the fourth-root of a fraction. He also gives a method (indirect, and requiring the solution of a biquadratic) for deducing the elements from the two arbitrary quantities. Mr. J. C. Glashan states that Mr. Young informed him of this discovery in the latter part of 1883, at which time my paper just quoted had not appeared.

48. Since the element-formulæ (7) enable us to express the elements in terms of the constituents, the problem of solving any quintic is reduced to that of determining its constituents. For the general quintic this can only be done by the aid of a sextic resolvent: For such individual quintics as exhibit certain relations between the coefficients the task is easier, and the quintics are said to be solvable. I say "said to be solvable," for the word solvable has received no exact definition. The knowledge of a value of v or of t is enough to effect a solution. This knowledge must be obtained legitimately. We have no right, for example, to construct a quintic from arbitrary values of the constituents, as in the last paragraph, and then declare it solvable because we know what the values are. If we had that right, we might determine the constituents of any quintic by means of a resolvent, then reconstruct the quintic from the values of the constituents so obtained, and thereupon call it solvable. The same quintic cannot be solvable to one inquirer and non-solvable to another, although the first may have constructed it from arbitrary constituents of which the second does not possess the secret. Wherein, then, does the distinction lie?

49. Shall we say that the formulæ (108) and (111), which express ε and ζ in terms of the four constituents, are criteria of solvability, and that all quintics are solvable whose coefficients are of that form? Then every quintic would be solvable, for all are of that form. Shall we limit the statement to cases wherein the constituents are rational? On the one hand no good reason can be given for such a limitation, and on the other hand it might be urged that the boundary is still too extended. Where else can the line be drawn? I am unable to answer, and think that the use of the word "solvable," so far as it implies a distinction between two supposed classes of solvable and insolvable quintics, should be discontinued. To justify this opinion, I shall take up several cases wherein solvability is conceded, and proceed to others between which and the first no dividing line can be established.

50. Any equation is solvable if any relation is known between its roots. If it be ascertained, for example, that the function of roots known as v, or that known as t, has a given value, the quintic is solvable. If in (50) $v = \gamma^2$ and $\delta = 0$, that equation becomes reduced to $4\gamma^2 - \varepsilon = 0$. Therefore, $y^5 + 10\gamma y^3 + 20\gamma^2 y + \zeta = 0$ is solvable, since we know that $v = \gamma^2$, nor is it necessary that either γ or γ^2 be rational. This is the oldest of the solvable quintics, and was discovered by DeMoivre. If in (50) $v = \gamma^2$, without restricting the value of δ, we have the relation between the coefficients shown in (19), or if $v = 0$, we have

(24). Whenever we meet with a quintic whose coefficients are observed to be related in either of these modes, we know the value of v, and the equation is solvable. Let us go a step further. If in (59) we put $s = 0$, we have as the criterion for a new class of solvable quintics,

$$\zeta l_1 = 25c_3^2. \tag{115}$$

Whenever this relation is observed to exist between the coefficients, we know that $s = 0$ and (unless $v = 0$) $t = 0$, and the quintic is solvable. It would be possible to form a table of criteria to any desired extent, for given successive values of t or v, so that the number of solvable classes is unlimited. But the argument can best be enforced by an illustration.

51. Let $t = -2$, and for simplicity let $\gamma = 0$. Then (4) becomes

$$\left.\begin{aligned} &1024\delta^4 + 128\delta\varepsilon\zeta - 64\varepsilon^3 + 32\delta\zeta^2 - 1408\delta^3\varepsilon - 32\varepsilon^2\zeta + 1200\delta^2\varepsilon^2 - 800\delta^3\zeta \\ &\quad + 320\delta^5 - 320\delta\varepsilon^3 + 240\delta^2\varepsilon\zeta + 60\delta^2\zeta^2 - 220\delta^4\varepsilon - 120\delta\varepsilon^2\zeta + 80\varepsilon^4 \\ &\quad + 6\delta\varepsilon\zeta^2 - 40\delta^3\varepsilon^2 + 118\delta^4\zeta - 8\varepsilon^3\zeta + 25\delta^6 - 6\delta^3\varepsilon\zeta - \delta\zeta^3 + \varepsilon^2\zeta^2 = 0. \end{aligned}\right\} \tag{116}$$

I say that any quintic whose coefficients are connected in this manner, when $\gamma = 0$, is solvable, because such coefficients indicate a known relation between the roots, namely, $t = -2$. In every such case, by (5),

$$v = -\frac{1}{25}\,\frac{-18\delta^4 + 4\delta\varepsilon\zeta - 2\varepsilon^3 + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta}{8\delta^2 - 4\delta\varepsilon - 2\delta\zeta + 2\varepsilon^2 - \delta^3}. \tag{117}$$

Then the elements of the roots are known by (7). We have here a "class of solvable quintics," with a suitable criterion, and an immediate solution. For instance, if we meet the equation $y^5 + 10y^2 - 10y + 51 = 0$, and test it by the criterion (116), we find the latter satisfied, and by (117) we find $v = 1$, so that by means of (7) it is at once solved. We might take any value of t, or any value of v, no matter why chosen or how determined, and obtain for it an equation like (116), to be thereafter regarded as a criterion for a "class of solvable quintics." But if any equation is "solvable" after such a criterion is laid down, it must be "solvable" before. I am unable, therefore, to concur in the usual assumption that some quintics are intrinsically solvable and that all others are not. Solution in every case depends on the recognition of the quintic as one of a previously discussed class, by observation of certain relations between its coefficients. The number of such possible classes is not limited. "An infinitely large accumulation of solvable quintics would leave none of the other sort, by which to illustrate the fact that the quintic cannot be solved algebraically; yet it would not change the fact."*

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* Vol. VI, p. 313.

52. Quintics "solvable" by peculiar methods may be brought under the general theory now presented. Reciprocal equations form a class of such quintics, their form being

$$ax^5 + bx^4 + cx^3 \pm cx^2 \pm bx \pm a = 0. \tag{118}$$

Comparing (52) and (53), we find that in this case $\phi = \mp \psi$, whence by (60) $\tau = \mp 1$, and by (61) $t = \mp 1 + ba^{-1}$. The solution of each such equation follows at once by (97) and (7). For example, if

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0, \tag{119}$$

we know that the roots are -1, $\frac{1 \pm \surd(-3)}{2}$, and $\frac{-1 \pm \surd(-3)}{2}$. By the present theory, we should proceed as follows to discover these roots. Since $\tau = -1$, $t = -1 + ba^{-1} = -\frac{4}{5}$. This value substituted in (97) gives $v^2 - \frac{13}{2} 5^{-3} v + \frac{1}{16} 5^{-4} = 0$, and the same result is obtainable from (95), or from the "second quadratic" of paragraph 40. Hence $v = \frac{13}{500} \pm \frac{12}{500}$. Taking $v = \frac{1}{20}$, we have $r_1 = -104.5^{-5}$, $r_2 = -4.5^{-3}\surd 5$, $s_1 = \frac{39}{2} 5^{-6}$, $s_2 = -\frac{237}{2} 5^{-8} \surd 5$, and employing these values in (7),

$$\left.\begin{aligned}
5^5u_1^5 &= -26 - 25\surd 5 + \tfrac{5}{2}\surd(1950 - 474\surd 5), \\
5^5u_2^5 &= -26 + 25\surd 5 + \tfrac{5}{2}\surd(1950 + 474\surd 5), \\
5^5u_3^5 &= -26 + 25\surd 5 - \tfrac{5}{2}\surd(1950 + 474\surd 5), \\
5^5u_4^5 &= -26 - 25\surd 5 - \tfrac{5}{2}\surd(1950 - 474\surd 5).
\end{aligned}\right\} \tag{120}$$

From this, by (6), the real root of the reciprocal equation (119) is

$$\left.\begin{aligned}
x_1 = u_1 + u_2 + u_3 + u_4 - \tfrac{1}{5} &= -0.2977659 + 0.5566772 \\
&- 0.5094635 - 0.5494478 - 0.2 = -1.
\end{aligned}\right\} \tag{121}$$

The other four values of v are $0.026 \pm 0.016\surd(-3)$ and $-0.022 \pm 0.008\surd(-3)$.

53. It is evident from (5) and (106), as well as from their mutual relations as functions of the roots, that v is rational whenever t is rational, and *vice versa*, assuming the coefficients rational. An apparent exception to this rule is found in some cases wherein v or t has two equal values, t or v having irrational values corresponding. In any such case, (5) or (106) becomes indeterminate, and recourse must be had, as in the preceding paragraph, and in other cases when t or v has equal values, to the equations (95) and (97). In the exceptional cases here spoken of, two equal values, say m, are to be regarded as a pair of irrational values, say $m \pm \surd n$, wherein n vanishes. Thus, if the roots be denoted by a, b, c, $d + \surd e$, $d - \surd e$, v (not t) will have two equal rational values if $e = d^2 - 2ad + a(b + c) - bc$. For instance, if the roots be -29, 21, -4, $6 + \surd(-25)$, $6 - \surd(-25)$, v has two equal values, namely, -125, which we may

regard as $-125 \pm \surd 0$, and the corresponding values of t are $-29 \pm \surd(-25)$. By referring to the definition in (38) of v as a function of the roots, it will be found that all of its values are real when all five roots are real, that some of its values are real when only one root is real, and that when three roots are real and unequal all of the six values of v are unreal, unless two are equal as just explained.

54. With a single exception, all efforts to resolve the quintic algebraically, with which I have become acquainted, have belonged to what I have called the first class, and have had for their object to illustrate the general theory of equations by determining the quantities required under that theory which I have here called the elements. The object is theoretic rather than practical, and when it is attained we have merely the satisfaction of knowing that we have accomplished for the fifth degree all that is possible under the uniform theory which affords solutions for the inferior degrees. I presume that the determination of the elements will continue to be regarded as the object of greatest interest in the resolution of the quintic. Nevertheless, it is not mentioned as essential in the definition quoted in paragraph 1, wherein the problem is broadly stated as the expression of the roots of the quintic in terms of those of a resolvent sextic. It is possible to accomplish this without reference to the elements, by some method of what I have called the second class. This view of the problem has, in the exceptional case referred to, been acted upon by a writer of the highest authority, Professor Cayley. I quote his statement* in the paragraph now following.

55. "The roots of the given quintic equation are each of them rational functions of the roots [ϕ] of the auxiliary equation, so that the theory of the solution of an equation of the fifth order appears to be now carried to its extreme limit. We have in fact

$$\begin{aligned}
\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5 &= (*\between x_1,\ 1)^4, \\
\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6 &= (*\between x_2,\ 1)^4, \\
\phi_1\phi_5 + \phi_2\phi_3 + \phi_4\phi_6 &= (*\between x_3,\ 1)^4, \\
\phi_1\phi_3 + \phi_2\phi_6 + \phi_4\phi_5 &= (*\between x_4,\ 1)^4, \\
\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6 &= (*\between x_5,\ 1)^4,
\end{aligned}$$

where $(*\between x_1,\ 1)^4$, etc., are the values, corresponding to the roots, x_1, etc., of the given equation, of a given quartic function. And combining these equations respectively with the quintic equations satisfied by the roots x_1, etc., respectively, it follows that, conversely, the roots x_1, x_2, etc., are rational functions of the combinations $\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5$, $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$, etc., respectively, of the roots of the auxiliary equation. It is proper to notice that, combining together in every possible manner the six roots of the auxiliary equation, there are in all fifteen combinations of the form $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$. But the combinations occurring in the above-mentioned equations are a completely determinate set of five combinations: the equation of the order 15, whereon depend the combinations $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$, is not rationally decomposable into three quintic equations, but only into a quintic equation having for its roots the above-mentioned five combinations, and into an equation of the tenth order, having for its roots the other ten combinations, and being an irreducible equation. Suppose that the auxiliary equation and its roots are known; the method of ascertaining what combinations of roots correspond to the roots of the quintic equation would be to find the rational quintic factor of the equation of the fifteenth order, and observe what combinations of the roots of the auxiliary equation are also roots of this quintic factor." Here, as Mr. Cayley states elsewhere,

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* "On a New Auxiliary Equation," Philosophical Transactions, Vol. 151, p. 264.

$$(*\between x_1, 1)^4 = 20(2a^2, 8ab, 22ac - 10b^2, 18ad - 10bc, 7ae - 10bd + 5c^2\between x_1, 1)^4.$$

56. The method just described requires for determining each root the ascertainment of the common root of a quartic and a quintic, and the process suggested for selecting fit combinations involves considerable difficulty. Two other methods of the second class will now be presented as alternatives.

57. If the several values of ϕ as defined in paragraph 15 be added two and two, there results the following scheme:

$$\left.\begin{array}{ll}
\phi_1 + \phi_2 = 2(x_1 - x_3)(x_5 - x_4), & \phi_2 + \phi_6 = 2(x_1 - x_5)(x_3 - x_2), \\
\phi_1 + \phi_3 = 2(x_1 - x_2)(x_5 - x_3), & \phi_3 + \phi_4 = 2(x_1 - x_5)(x_4 - x_3), \\
\phi_1 + \phi_4 = 2(x_1 - x_4)(x_2 - x_3), & \phi_3 + \phi_5 = 2(x_2 - x_3)(x_4 - x_5), \\
\phi_1 + \phi_5 = 2(x_1 - x_5)(x_2 - x_4), & \phi_3 + \phi_6 = 2(x_1 - x_3)(x_4 - x_2), \\
\phi_1 + \phi_6 = 2(x_2 - x_5)(x_3 - x_4), & \phi_4 + \phi_5 = 2(x_1 - x_3)(x_2 - x_5), \\
\phi_2 + \phi_3 = 2(x_1 - x_4)(x_5 - x_2), & \phi_4 + \phi_6 = 2(x_1 - x_2)(x_4 - x_5), \\
\phi_2 + \phi_4 = 2(x_2 - x_4)(x_5 - x_3), & \phi_5 + \phi_6 = 2(x_1 - x_4)(x_3 - x_5). \\
\phi_2 + \phi_5 = 2(x_1 - x_2)(x_3 - x_4), &
\end{array}\right\} \quad (122)$$

In these products of differences each difference occurs three times, each time with different values of ϕ. If any pair of values of ϕ be taken at random, the remaining four values can be arranged as different pairs in three sets, two of which will exhibit common factors with the first pair, while the third will not. Let any three pairs having common factors be represented by $p_1 = 2(x_2 - x_3)(x_4 - x_5)$, $p_2 = 2(x_3 - x_1)(x_4 - x_5)$, and $p_3 = 2(x_1 - x_2)(x_4 - x_5)$. Then

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = 0. \tag{123}$$

With the three quintics in x_1, x_2, and x_3 respectively, we have now four equations from which the values of these roots may be obtained by elimination. Once in three times, as has been seen, this process will fail, in which event the use of another combination will be successful. The method here given is simpler in appearance than that of Mr. Cayley, but it is open to the same objections, that it is only tentative, and that it is so laborious as to be unavailable in practice. The second method, now to be given, will be found free from both objections.

58. If we add two and two the several values of ψ as defined in (44), and compare the results with (122), we find that

$$\left.\begin{array}{ll}
\psi_1 + \psi_6 = x_1(\phi_1 + \phi_6), & \psi_4 + \psi_6 = x_3(\phi_4 + \phi_6), \\
\psi_2 + \psi_4 = x_1(\phi_2 + \phi_4), & \psi_1 + \psi_3 = x_4(\phi_1 + \phi_3), \\
\psi_3 + \psi_5 = x_1(\phi_3 + \phi_5), & \psi_2 + \psi_6 = x_4(\phi_2 + \phi_6), \\
\psi_1 + \psi_2 = x_2(\phi_1 + \phi_2), & \psi_4 + \psi_5 = x_4(\phi_4 + \phi_5), \\
\psi_3 + \psi_4 = x_2(\phi_3 + \phi_4), & \psi_1 + \psi_4 = x_5(\phi_1 + \phi_4), \\
\psi_5 + \psi_6 = x_2(\phi_5 + \phi_6), & \psi_2 + \psi_5 = x_5(\phi_2 + \phi_5), \\
\psi_1 + \psi_5 = x_3(\phi_1 + \phi_5), & \psi_3 + \psi_6 = x_5(\phi_3 + \phi_6). \\
\psi_2 + \psi_3 = x_3(\phi_2 + \phi_3), &
\end{array}\right\} \tag{124}$$

Hence any pair of values of ψ divided by the corresponding pair of values of ϕ gives a root of the quintic; and the five pairs formed by combining any value of ψ with the other five values respectively, each pair being similarly divided, give all the roots. It is to be remembered that $\psi = \tau\phi$, where $\tau = t - ba^{-1}$, that $\phi^2 = 500v$, and that $x = y - ba^{-1}$. The values of y are therefore to be found by dividing pairs of values of $t\phi$ or $tv^{\frac{1}{2}}$ by the corresponding pairs of values of ϕ or $v^{\frac{1}{2}}$ respectively. In practice, there are two modes of determining the needed values. We may determine the values of ϕ by the dexter resolvent, and from these the corresponding values of t by (106); or, in many cases preferably, we may ascertain the values of t by the central resolvent, and employ them in (5) to obtain those of v. The signs to be given to the several values of $v^{\frac{1}{2}}$ or ϕ in the latter case must be such as to agree with $\Sigma v^{\frac{1}{2}} = \Sigma\phi = 0$, and if more than one such arrangement is possible, with $\Sigma t v^{\frac{1}{2}} = \Sigma t\phi = 0$. In the highly improbable event that more than one arrangement fulfils both conditions, recourse may be had to (49) or (48).—As an illustration, let us take (12), for which the several values are

$$\left.\begin{array}{lrrrrrr} t\phi & 150 & -174 & 130 & -66 & -22 & -18 \\ \phi & 5 & -29 & 51 & -19 & -69 & 61 \end{array}\right\} \qquad (125)$$

Here the sum of any two values of $t\phi$, divided by the sum of the corresponding values of ϕ, gives a value of y, which increased by 1, the value of $-ba^{-1}$, gives a root of the quintic. If any one value of $t\phi$ be added successively to each of the others, and the sums be separately divided by the corresponding pairs of values of ϕ, we shall obtain all five values of y, and, by adding 1 to each, all five roots of the given quintic.

59. Various transcendental solutions of the quintic, by the aid of elliptic functions or of hypergeometric series, have been proposed in recent years. It would seem that no solution involving series can rival in simplicity the use of Lagrange's theorem. The employment of elliptic functions appears to have been suggested by De Moivre's transcendental solutions, by circular functions, of the cubic and of the solvable quintic known by his name. The use of tabulated functions has long been recognized as legitimate in analysis whenever necessary, with the implied understanding that no new function is to be introduced for tabulation so long as those already recognized will serve. Solutions by elliptic functions are circuitous and difficult, yet we are bound to prefer them to others which would require new tabulations. Were it not for this restriction, we might devise solutions of the quintic with great ease. Two such possible solutions will now be shown, in each of which it is assumed that the quintic has been brought by Jerrard's transformation to the form $x^5 + 5\varepsilon x + \zeta = 0$.

60. In the case just mentioned, if we put $v = \frac{1}{25}\varepsilon z$, the dexter resolvent (50) becomes, with slight changes,

$$(z-1)^4(z^2 - 6z + 25) = z\varepsilon^{-5}\zeta^4. \qquad (126)$$

"In this compact equation we have the unknown quantity expressed directly as a function of the parameter $\varepsilon^{-5}\zeta^4$. Were tables for such purposes worth calculating, a table of single entry might be constructed with great ease, showing a value of the parameter for each value of z, and therefore the converse, which would supply all that is needed to exhibit the roots of the trinomial in their normal form."*

61. A second possible solution involving special tabulation will be found to follow from consideration of De Moivre's solution of the cubic. If $x = \cos\theta$

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* Vol. VI, p. 307.

$= \frac{1}{2}(\exp[i\theta] - \exp[-i\theta])$, where $i = \sqrt{(-1)}$,

$$x^3 - \tfrac{3}{4}x - \tfrac{1}{4}\cos 3\theta = 0. \tag{127}$$

To quote De Morgan's remark on the Irreducible Case,* "the best method of obtaining the roots is by having recourse to a registry of the roots of cubic equations, which is in the hands of every tyro, namely, the tables of sines and cosines." In other words, if a table of cosines (or sines) were needed for no other purpose, it would be useful at any rate as a registry of the roots of cubic equations. As a table of roots, however, it would be incomplete, since this method serves only when there are three real roots, and it would need to be supplemented, were no other solution known, by the values of $\exp\theta + \exp(-\theta)$. For quintics a similar "registry of roots" might be prepared, which would likewise serve only for roots all real, and would therefore need to be similarly supplemented. The function to be tabulated would be $\cos\theta + 5\cos\frac{3}{5}\theta$, which let us denote by $C\theta$. Then

$$x^5 - \tfrac{5}{8}x - \tfrac{1}{16}C5\theta = 0 \tag{128}$$

The trinomial to be solved having been first reduced to the form $x^5 - \frac{5}{8}x - k = 0$, the value of 5θ would be found from the new table, and, as before, $x = \cos\theta$. Or, we might have $x = \sin\theta$, and tabulate $S\theta = \sin\theta - 5\sin\frac{3}{5}\theta$, when the equation would be

$$x^5 - \tfrac{5}{8}x - \tfrac{1}{16}S5\theta = 0. \tag{129}$$

62. I close with the suggestion that the solution of the special Jacobian sextic discussed by MM. Hermite, Kronecker, Brioschi, Kiepert and others (and therefore, as shown by Signor Brioschi,† of the Malfattian or dexter resolvent), by means of elliptic functions, should be combined with the methods of exhibiting the roots now shown in paragraphs 2 and 58.

MILWAUKEE, August 5, 1885.

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* Penny Cyclopædia.

† Mathematische Annalen, XIII, 154.

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